Cigarettes

↓

Theory
OUTSIDE LOOK: 65 NOTEBOOKS
VERY STRONG “FLAVORS”

✓ Cigarette smoke
✓ Pen ink
CONTENTS

All academic: evolution of ideas, development of techniques

- Notes/Summaries for talks, lectures, courses, ...
- Drafts for papers, book chapters, preprints, ...
- Work summary
- Maple calculations (symbolic, numerical, figures, tables, expansions)
- Email/letter correspondences
- Miscellaneous

Three categories

- Finished, well-explored techniques/topics
- Unfinished
- We-don’t-know-yet
AGENDA 0 & CAHIER LXIV
Les nombres de Stirling de seconde espèce : série ordinaire et exponentielle

$S_{n,k}$ désigne le nombre de partitions de $[n]$ en $k$ classes (blocs) ;
$k! S_{n,k}$ représente ainsi le nombre de surjections de $[n]$ sur $[k]$.

Les deux expressions de série génératrices (ordinaires ou exponentielles)
de $S_{n,k}$ correspondent à deux présentations différentes de partitions
d’ensembles.

1) $\sum S_{n,k} \frac{x^n}{n!} = \left(\frac{e^x - 1}{x}\right)^k$

2) $\sum S_{n,k} \frac{3^n}{k^n} = \frac{3^k}{(1-3)(1-23)\ldots(1-k3)}$

Une preuve directe de ces deux séries génératrices correspond à une preuve
combinatoire que 2) se transformer de la place - Béal de 1.

Pour 1), la preuve résulte de manipulations élémentaires de séries génératrices
associées à des mots (cf Analyse d’Algorithmes 1980).
SNAPSHOTS

LE POINT DES ARTICLES

Some Unix Tricks & Features

(a) echo 'who | wc -e' | users logged in

(b) for i in 1 2 3 4
    do
done

(c) for data in "0.1 2.0" "0.2 4.5" "0.3 4.25"
do
    date >> tmp
done

(d) echo $data >> results

(e) for i in 1 2 3
    do
cat done < toto
    done

(f) sh (m) en shell?

(g) backup

(h) shotty all

(i) & echo 'yes! $2 is there'

(j) cat $1

(k) & echo 'expr $a + 1'

(l) & echo 'expr $a + 1'

(m) cat ...
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<th>Rank</th>
<th>( \epsilon )</th>
<th>( \gamma )</th>
<th>( \delta )</th>
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<td>8.76</td>
<td>22.03</td>
<td>50.61</td>
<td>112.19</td>
<td>244.87</td>
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</table>
SNAPSHOTS

> bernoulli[d] // Finds the eigenfunction of degree d for G-uniform
f := proc(d) option remember; local p, i, a;
p := x^d+sum(a[i]*x^i, i = 0 .. d-1); expand(p^2*(d-1)*subs(x=x/2, p)+subs(x=(x+1)/2, p));
end;

> seq(bernoulli(d), d=0..10);
seq(f(), i=0..10);

f :=
proc(d)
local p, i, a;
options remember;
p := x^d+sum(a[i]*x^i, i = 0 .. d-1);
expand(p^2*(d-1)*subs(x = 1/2*x, p)+subs(x = 1/2*x+1/2, p));
end;

> seq(coeff(f, x^i), i = 0 .. d-1);
solve(f, seq(a[i], i = 0 .. d-1));
subs(*, p);
end;

> seq(bernoulli(d), d=0..10);
seq(f(), i=0..10);

> bernoulli[11, x];

\[\frac{x^{10} + 10x^9 + 45x^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2 + 10x + 1}{12},\]

\[\frac{x^{10} - 10x^9 + 45x^8 - 120x^7 + 210x^6 - 252x^5 + 210x^4 - 120x^3 + 45x^2 - 10x + 1}{12}\]
collé assez bien à l'analyse. Je fais environ 1000 000 de simulations à la minute de l'algorithme de Gauss. Voici un tableau comparatif entre les probabilités estimées d'un précédent message et les probabilités observées sur un tirage d'un million. On doit pouvoir faire 100 millions les doigts dans le nez, et gagner un digit de précision supplémentaire. La conclusion irremédiable est donc en tout cas que "mon générateur aléatoire est bon"!

NB: Il s'agit des probabilités cumulées, P(\overline{E}^n+h).

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<th>( m )</th>
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<th>Probabilité observée</th>
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<tr>
<td>10</td>
<td>0.000001</td>
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</tbody>
</table>

\[ \text{Tr}(g_n g_n) = \sum_{m=0}^{\infty} \bar{m}^2 \quad \text{avec} \quad \bar{m} = [m_0, m_1, \ldots] \]

\[ \text{Tr}(g_n + g_n) = \sum_{m=0}^{\infty} \bar{m}^4 \]

and then

\[ \text{Tr}(g_n) = \sum_{m=0}^{\infty} \frac{\bar{m}^2}{n^2} \]

By the same principle,

\[ \text{Tr}(g_n^4) = \sum_{m_0}^{\infty} \frac{\bar{m}^4}{(n^2 + m^2)^2} \]

Ref. D. Mayer & A. Tissé formule de Bateman. (9)

Then the power sum can be computed.

\[ S_n = \sum_{\lambda=1}^{\infty} \lambda \nu^{x} , \quad x = 4 \]

\[ S_2 = \sum_{\nu=1}^{\infty} \nu^2 \nu = \sum_{\nu=1}^{\infty} \frac{\nu^3}{\nu^2} \]

\[ = \Phi(2) \]

where \( \Phi(2) = \mathcal{M}(\nu, 2 \lambda_1) \) is an entire function by result from Ruelle (1), Babenko, etc.
\[ T(z) = \frac{1}{z} - \left[ \text{Re} \left( \frac{1}{z} \right) \right] \]

\[ L_n(z) = -n \log \frac{z}{e} - \frac{n^2}{2} - \log e \cdot \log(1-z) \]

\[ L_2(e^{\frac{k}{n}}) = C \cdot (1-2) \]

\[ E = \sum_{n=1}^{\infty} \frac{B_n}{n} \left( t^{1-2} \right) \]

\[ C = \sum_{n=1}^{\infty} \frac{B_n}{n} \left( \frac{1}{n+2} \right) = 0.35333 \]

\[ D = \sum_{n=1}^{\infty} \frac{B_n}{n} \left( \frac{1}{n+2} \right)^2 = 0.1961970 \] (Check with numerical integration)
Recent decades have seen a surge of interest in discrete mathematics and combinatorics, where what is at stake is the study of properties of finite objects constructed by a finite set of rules. Structural combinatorics is the most visible part of the iceberg in Amdouni, with markedly profound result, such as the theory of minors or the long-sought solution to the perfect graph conjecture. The problem there is to answer questions such as: Given such and such construction rules, which properties must the corresponding objects invariably satisfy?8

Partly pushed by the needs of several branches of science—e.g., computer science to probability theory to bio-information to statistical physics—the past twenty to thirty years have seen the appearance of a large number of studies dedicated to another (at least equally) fundamental question. Given such and such construction rules, which properties must the corresponding objects satisfy in an overwhelming proportion of cases? In other words, rather than seeking what is always true, we want to characterize what is almost always true. In this discrete combinatorial world, we want to measure things.

My own research since the early 1990s has been precisely a sustained effort meant to address the characterization of what is “almost always” true amongst the most important structures of mathematics, such as words, trees, mappings, graphs, and permutations. It has constantly alternated between methodological work (e.g., the development of singularity analysis with Odlyzko; the elucidation of the power of the Mellin transform) and research dedicated to solving concrete problems arising from applications (e.g., the coalescence of hashing tables and the universality of Airy laws, the analysis of sequences in relation to probabilistic counting algorithms).

An outcome is the publication by Cambridge University Press (at the turn of 2006-2009) of an 825 pages monograph coauthored with Sedgwick and titled Analytic Combinatorics, which I will abbreviate as AC.

The field of analytic combinatorics as expounded in the book AC constitutes the basic layer on which the present proposal is built. Roughly, the main theme is that a class of combinatorial structures is reduced to a locally smooth surface (the Riemann surface of a corresponding generating function), whose “critical” (the distinguished maxima and minima) are seen to contain a host of quantitative information. For instance, as I showed with Odlyzko in 1982 and 1994, the complex-analytic structure of the iteration \( y_n = z + \frac{1}{y_{n-1}} \) near

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8In December 2008, right before the release of AC, qualified some as “the best-seller in American statistics,” the Google Scholar citation score of the manuscript (which had been on the web for several years) was 240, including self-citations. This appears to be substantially more than the average citation score of an existing mathematics book.
On a problem of Wilf

(De Weger et al., JCTA, 1995)

Define $D_n = \sum (-1)^n \left(\frac{n}{2}\right)$ and set $\mu = \exp(1-e^2)$.

Computation $D_n = 0$ for all $n \geq 2$.

$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

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$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$
Stirling #s of the 2nd kind
Symbolic method in analysis of tree algorithms
Remarks on partitions
Trie statistics
q-Laguerre polynomials
Simon-Newcomb problem
Lexicographic tree height
Search tree height
Approximate counting
Complexity calculus
Markov chains
Exp-variate generation

Talk by Guy Fayolle
Extension of approximate counting
Asymptotics/Mellin
Distribution of path areas
Pippenger’s communication protocols
Combinatorial sums asymptotics
DST asymptotics
Grid file algorithms
Functional graphs
Differential equations & linear systems
Random mappings from a finite set into itself are either a heuristic or an exact model for a variety of applications in random number generation, computational number theory, cryptography, and the analysis of algorithms at large. This paper introduces a general framework in which the analysis of about twenty characteristic parameters of random mappings is carried out: These parameters are studied systematically through the use of generating functions and singularity analysis. In particular, an open problem of Knuth is solved, namely that of finding the expected diameter of a random mapping. The same approach is applicable to a larger class of discrete combinatorial models and possibilities of automated analysis using symbolic manipulation systems (“computer algebra”) are also briefly discussed.

Initiated in 1982 ➔ Published in 1990
**DEEPER LOOK**

Flajolet, Knuth & Pittel (1989)  
The first cycles in an evolving graph

- Backhouse’s constant
- Gray code function
The following problem was mentioned as open (and due to Erdős) at the Random Graph Conference in Poznań (Aug. 1985) by Michael Karoński.

Take a set of \( n \) nodes (disconnected, i.e. \( K_0 \)) and

Draw in edges at random until a circuit occurs. How large is the circuit?

First approach: Determining stopping times.

Notations: \( n \) = # of points \( N = \binom{n}{2} \) (there are no self-loops).

Points are assigned to be labeled from 1 to \( n \) by distinct integers.

A configuration (unordered reduced) of size \( n \) is formed of:

- a cycle of trees with one edge on the cycle marked
- a set of disconnected trees (node disjoint)

A cycle has length \( \geq 3 \), trees are not rooted a cycle is not oriented.

The process of edge along each of the possible edges is equally likely chosen.

- Depth 2
- Depth 4
- Depth 2
- Depth 3

\( N \) choice, \( \binom{N-1}{2} \) choice, \( \binom{N-2}{2} \) choice, \( \binom{N-3}{2} \) choice.
FIRST CYCLE IN EVOLVING GRAPHS

An extended (edge time-changed) configuration is a reduced configuration with edges labelled in order $3, 0, 3, \cdots$, the marked edge having the largest label.

Claims: (1) Each terminal node of the process tree is described by an extended configuration.
(2) Each extended configuration at depth $k$ has probability
$$ \frac{1}{N(N-1)(N-K+1)} $$
(3) To each standard configuration, there correspond exactly $(k-1)!$ extended configurations, each equally likely.

Thus:

Lemma: If $C_{n,k}$ is the number of standard configurations as a node with $k$ edges, then
$$ P_{n,k} = \frac{(k-1)!}{N(N-1)(N-K+1)} C_{n,k} $$

And we have the shape form:

Lemma: Let $G$ be a property of standard configurations, $P_{n,k}(G)$ the number that the process stops with $G$ satisfied, $C_{n,k}(G)$ the number of standard configs satisfying $G$, then:
$$ P_{n,k}(G) = \frac{(k-1)!}{N(N-1)(N-K+1)} C_{n,k}(G) $$

and
$$ P_n(G) = \frac{1}{N} \sum_{k=1}^{n-1} \left( \frac{N-1}{k-1} \right)^{-1} C_{n,k}(G) $$


\[ \begin{array}{l}
\text{Verifical: } \quad n=3 \\
\quad N=3 \\
\quad \text{5 standard config} \\
\quad P_{3,3} = \frac{2}{3} \\
\text{3 @ 1} \\
\end{array} \]

\[ \begin{array}{l}
\text{6 @ 4} \\
\text{9 @ 6} \\
\end{array} \]

\[ \begin{array}{l}
\text{3 @ 1} \\
\text{5 @ 5} \\
\text{8 @ 8} \\
\text{1 @ 1} \\
\end{array} \]

Now consider edges: observe that in each tree
$$ \# \text{ edges} = \# \text{ nodes} - 1 $$

Counting the number of standard configurations (e.g., gen fn)

let $Y(t) = \text{gen fn}$
$Y(t) = 2^{\frac{m-1}{m}}$ for gen fn for rooted trees
$Y(t) = 2^{\frac{k}{m}}$ for unrooted trees

Then
$$ C(t) = \frac{1}{2} \left( 1 - Y(t) \right) e^{\frac{3}{2}} $$

\[ \text{once } \frac{m}{2} \text{ with the cycle condition} \]
\[ \text{a marked end = an open cycle} \]
\[ \text{a list of trees} \]
\[ \text{e.g., a bunch of trees} \]
FIRST CYCLE IN EVOLVING GRAPHS

Thus
\[ \Sigma C_{w,k} u^k b^w = \frac{1}{2} \sum_{n=1}^{\infty} \gamma^{(2n)}(u) \frac{d^2}{du^2} \gamma^{(2n)}(u) \]

But, as remarked by J.W. Nunn, \( \gamma(t) = \gamma(t) - \frac{1}{2} \gamma^{(2)}(t) \), then
\[ C(u,\beta) = \frac{d^2}{du^2} \gamma^{(2n)}(u) = \frac{d^2}{du^2} \gamma^{(2n)}(u) \]

While we have already seen, as
\[ \frac{2}{3} \int_{\xi}^{\infty} \int_{\xi}^{\infty} (12u^3 + 6u^2) = O(3) \]

Continuation: Some random ideas

A variant of Lagrange's formula should be useful.
\[ \int_0^1 F(Y) \frac{d^2}{dY^2} = \int_0^1 F(Y)(H-Y) \gamma^{(2)} \gamma^{(2)} \gamma^{(2)} Y^{(2n)} e^{-Y} \]

\[ \left[ \frac{1}{2} \right] F(Y) = [Y^n] F(Y)(H-Y) e^{-Y} \]

How to do the probability weighting?

Let \( A(3, u) \geq a_n u^k b^n \). Then
\[ A(3, u) \otimes C_g(3) = \sum_{n=1}^{\infty} a_n u^k b^n (1-u)^{2n} \]

and integrate from 0 to 1, we need
\[ \int_0^1 u^k (1-u)^{2n} \frac{d}{du} \left( \sum_{k=0}^{\infty} \frac{(u-1)^n}{u^n} \right) \]

That does it!!! Need, maybe, use technique à la Kneser-Feller.

leaves promising!! To be done!!!
FIRST CYCLE IN EVOLVING GRAPHS

Approximation & heuristics

\[ \text{Set } L = m - 3 - 1 \quad l = m - 3 - 2, \text{ then:} \]

\[ E_{m,n} = \sum_{l=0}^{m-3} \frac{(-1)^l}{L! L (k-3)l! (m-k-1)l!} \cdot \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \cdot \frac{1}{2} \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \]\n
\[ \sum_{l=0}^{m-3} \left( \frac{(-1)^l}{L! L (k-3)l! (m-k-1)l!} \right) \cdot \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \cdot \frac{1}{2} \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \]

\[ \approx \frac{4}{2} \sum_{k=3}^{m-3} \frac{(-1)^l}{L! L (k-3)l! (m-k-1)l!} \cdot \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \cdot \frac{1}{2} \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \]

\[ \approx \frac{4}{2} \sum_{k=3}^{m-3} \frac{(-1)^l}{L! L (k-3)l! (m-k-1)l!} \cdot \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \cdot \frac{1}{2} \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \]

If formula has a wider range, then:

\[ \frac{4}{2} \sum_{k=3}^{m-3} \frac{(-1)^l}{L! L (k-3)l! (m-k-1)l!} \cdot \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \cdot \frac{1}{2} \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \]

\[ \approx \frac{4}{2} \sum_{k=3}^{m-3} \frac{(-1)^l}{L! L (k-3)l! (m-k-1)l!} \cdot \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \cdot \frac{1}{2} \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \]

A more refined estimate should show that \[ \frac{4}{2} \sum_{k=3}^{m-3} \frac{(-1)^l}{L! L (k-3)l! (m-k-1)l!} \cdot \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \cdot \frac{1}{2} \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \]

Thus one should expect \[ E(Cycle) \sim Cm/n + o(f) \]

What next? What about a double saddle point argument? (invariant to \( m/n \))

\[ \sum_{k=3}^{m-3} \frac{(-1)^l}{L! L (k-3)l! (m-k-1)l!} \cdot \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \cdot \frac{1}{2} \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \]

\[ \approx \frac{4}{2} \sum_{k=3}^{m-3} \frac{(-1)^l}{L! L (k-3)l! (m-k-1)l!} \cdot \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \cdot \frac{1}{2} \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \]

\[ \approx \frac{4}{2} \sum_{k=3}^{m-3} \frac{(-1)^l}{L! L (k-3)l! (m-k-1)l!} \cdot \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \cdot \frac{1}{2} \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \]

\[ \approx \frac{4}{2} \sum_{k=3}^{m-3} \frac{(-1)^l}{L! L (k-3)l! (m-k-1)l!} \cdot \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \cdot \frac{1}{2} \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \]

\[ \approx \frac{4}{2} \sum_{k=3}^{m-3} \frac{(-1)^l}{L! L (k-3)l! (m-k-1)l!} \cdot \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \cdot \frac{1}{2} \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \]

\[ \approx \frac{4}{2} \sum_{k=3}^{m-3} \frac{(-1)^l}{L! L (k-3)l! (m-k-1)l!} \cdot \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \cdot \frac{1}{2} \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \]

\[ \approx \frac{4}{2} \sum_{k=3}^{m-3} \frac{(-1)^l}{L! L (k-3)l! (m-k-1)l!} \cdot \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \cdot \frac{1}{2} \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \]

\[ \approx \frac{4}{2} \sum_{k=3}^{m-3} \frac{(-1)^l}{L! L (k-3)l! (m-k-1)l!} \cdot \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \cdot \frac{1}{2} \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \]

\[ \approx \frac{4}{2} \sum_{k=3}^{m-3} \frac{(-1)^l}{L! L (k-3)l! (m-k-1)l!} \cdot \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \cdot \frac{1}{2} \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \]

\[ \approx \frac{4}{2} \sum_{k=3}^{m-3} \frac{(-1)^l}{L! L (k-3)l! (m-k-1)l!} \cdot \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \cdot \frac{1}{2} \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \]

\[ \approx \frac{4}{2} \sum_{k=3}^{m-3} \frac{(-1)^l}{L! L (k-3)l! (m-k-1)l!} \cdot \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \cdot \frac{1}{2} \left( \frac{1}{2} \right)^{n-k} \cdot \frac{1}{n-k}! \]
FIRST CYCLE IN EVOLVING GRAPHS

Summary. The main steps:

1. First cycle uses function kernel equation for the field of integration:
   \[ F_{i+1}(u, y) = \sum_{j} W_{ij} F_{j}(u, y) \]

2. For a fixed cycle, fixed # of nodes or site
   \[ \theta_{i}(y) = F_{i}(y) \]

3. For generation of interest, evaluate the integrated kernel by:
   \[ \theta_{i}(y) = \int_{0}^{t} F_{i}(u, y) \, du \]

4. Evaluate \( \theta_{i}(y) \) when \( u = 0, t \) to 0, then project
   \[ \theta_{i}(y) \] on G that is subject to
   \[ \theta_{i}(y) = \frac{1}{2} \int_{0}^{t} \phi_{i}(u, y) \, du \]

and take the proof of \( \theta_{i}(y) \) to \( \theta_{i}(y) \) to be obtained.
FIRST CYCLE IN EVOLVING GRAPHS

Note: Expect $\lambda \leq 1$ to give most of contribution since

$(-1)^n \left(\frac{m}{n}\right)^k$ is maximized when $n \approx \frac{\lambda}{m} \leq \frac{1}{m}$.

Note: things should happen when $\lambda = 1$.

Good hope: saddle point for integral with $\frac{1}{2} \frac{y^2}{1-y} e^{y}$ appears when $y = x$ which indicates that $\log y$ is exactly cancelled by the other one...! Every thing looks promising again.

Take for instance path that cycle has a hole in. Then try with $\frac{x}{\lambda} \int \frac{1}{1-y} e^{-y} dy = \frac{x}{\lambda} \log(1-y)$

but saddle point appears when $\frac{2}{\lambda} \log(1-y) = \frac{1}{\lambda} = 0 \Rightarrow$ take $y = 1$.

Then

$\int_{0}^{1} \frac{x}{\lambda} \log(1-y) e^{-y} dy
=rac{x}{\lambda} \left[ -\frac{1}{2} \lambda - \frac{1}{2} \lambda \right]
=-\frac{1}{2} \lambda e^{-1} - \frac{1}{2} \lambda e^{-1}
=rac{1}{2} \lambda e^{-1} - \frac{1}{2} \lambda e^{-1}

\phi(n) \sim \frac{e^{n/2}}{\sqrt{2\pi n}} \phi_{1}(n)

\text{Saddle points}

$\lambda < 1 \quad y = \lambda
\lambda = 1 \quad y = e^{-1/2}
\lambda > 1 \quad y = e^{-1/2}$

$E(K_n \sim x^2 e^{-x/2} \log^2 m x/2)$

WOW!!!

To be checked...

An effective bound for $\lambda \leq \frac{1}{2}$ on $B_n(n)$:

$B_n(n) = \frac{n}{2} e^{-n/2} \sqrt{2\pi n}

Take $\lambda - 1$ as cost of integration then

$B_n(n) \leq \frac{1}{2} e^{n/2} m$

Thus

$\int_{0}^{1} B_n(n) \frac{1}{2} e^{n/2} m$

small time $x^2$,

$\int_{0}^{1} B_n(n) \frac{1}{2} e^{n/2} m$

$= \lambda$. Uniformly in $\lambda$ (SS)

Observation: $B_n(n)$ decreases with $B_{n+1}(n)$ which increases. In general expected for the kernel $B_n(n)$.

The Saddle point image makes sense.

Last expression: further probably.
FIRST CYCLE IN EVOLVING GRAPHS

August 18, 1995

Some remarks on Oddly-Even Theorems

 Aim is to prove:

Theorem. Let $f(x)$ be analytic for $|x|<1$ with the exception of a zero of multiplicity at $x=1$. Then $f(x)$ satisfies $\frac{f(x)}{1-x}$.

Proof:

1. $L$ is an increasing function as $x \to \infty$
2. $L$ is slowly increasing, that is, $L(x) \sim x^\gamma$
3. $L(x) \to \frac{1}{\gamma}$ as $x \to \infty$

Then $\log x = O(\frac{1}{n} \log L(n))$.

Example:

- $[3^m] O(\frac{1}{m-3}) \log \log \frac{m}{1-3} \Rightarrow O(\alpha \log \log m)$
- $f(\infty) = 0$, $f(0) = \infty$

Fur reasons, it is MP for methods.

Further, the MP

- $L(\infty) = \frac{1}{\gamma}$

Then use the $MP$ method.

- $\int \limits_{r_1} \int \limits_{r_2} = O(\frac{1}{n} \log L(n))$
- $|\int \frac{1}{n} \frac{1}{n^2} \log L(n) = \frac{1}{n^2} \log L(n)$

Hence, it is the fact that $L$ is increasing.

Pattern: What about slowly decreasing function?
**FIRST CYCLE IN EVOLVING GRAPHS**

### Numerical Values

<table>
<thead>
<tr>
<th>m</th>
<th>n</th>
<th>Values of ( (1-u)^n ) En(( \frac{u}{1-u} ))</th>
<th>( \frac{1}{n} ) En(( \frac{u}{1-u} ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>10</td>
<td>0.857146286 2.8130 0.0738 2.31</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>15</td>
<td>0.87843 1.76125 0.0813 1.38</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>20</td>
<td>1.4760 1.464 0.1673 0.68</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Calculation

- \( E(\lambda n^{-3}) = \int_0^1 (1-u)^n En(\frac{u}{1-u}) \ du \)
- \( \text{where } En(B) = \sum_{k=0}^{\infty} \frac{B^k}{k!} \)

### Conclusion

- The result is based on 2 trials.
- 1st trial: 26.316 13.42
- 2nd trial: 26.316 13.42

### Notes

- Calculations subject to further refinement.
- The result holds for all values of \( \lambda \) due to saddle point effects.
FIRST CYCLE IN EVOLVING GRAPHS

Problem: (open) What is the distribution of \( \{X_0, X_1, X_2, \ldots \} \) in the sequence defined on \([0, 1]\)?

What should be the initial distribution? If it admits a density \( f(x) \) then once we are inside the interval, of \( f(x) \) we should have reason of "flow."

\[
\alpha(x) = \frac{v}{2} \alpha(x/2) \quad \text{if} \quad x > 1/2
\]

\[
\alpha(x) = \frac{v}{2} \alpha(x/2) + f(1-x/2) \quad \text{if} \quad x < 1/2
\]

Selon la répartition de \( \alpha(x) \) on trouve pour \( v=0.1, 0.2, 0.3, 0.4, 0.5 \):

<table>
<thead>
<tr>
<th>( n )</th>
<th>100</th>
<th>1000</th>
<th>10 000</th>
<th>100 000</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>45</td>
<td>152</td>
<td>205</td>
<td>266</td>
</tr>
<tr>
<td>2</td>
<td>67</td>
<td>144</td>
<td>217</td>
<td>283</td>
</tr>
<tr>
<td>3</td>
<td>92</td>
<td>102</td>
<td>285</td>
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</tr>
<tr>
<td>4</td>
<td>124</td>
<td>131</td>
<td>291</td>
<td>407</td>
</tr>
<tr>
<td>5</td>
<td>165</td>
<td>179</td>
<td>334</td>
<td>509</td>
</tr>
<tr>
<td>6</td>
<td>221</td>
<td>336</td>
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</tr>
<tr>
<td>7</td>
<td>319</td>
<td>358</td>
<td>483</td>
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</tr>
<tr>
<td>8</td>
<td>436</td>
<td>420</td>
<td>615</td>
<td>979</td>
</tr>
<tr>
<td>9</td>
<td>597</td>
<td>478</td>
<td>929</td>
<td>1253</td>
</tr>
<tr>
<td>10</td>
<td>717</td>
<td>558</td>
<td>1020</td>
<td>1261</td>
</tr>
</tbody>
</table>

\[ A(x) = \int_0^x \alpha(t) \, dt \]

\[ A(x) = A(2x/3) + A(2x/3) - A(x) \quad \text{if} \quad x > 1/2 \]

\[ A(x) = A(2x/3) + A(2x/3 - x) - A(2x/3) \quad \text{if} \quad x < 1/2 \]
FIRST CYCLE IN EVOLVING GRAPHS

Another approach: \( u_n \) has distribution \( D \) over \((0, 1)\)

\[ u_n \sim \left\{ \begin{array}{ll} \frac{1}{2} & \text{w.p. } \frac{1}{2} \\ \frac{1}{2} & \text{w.p. } \frac{1}{2} \end{array} \right. \]

\[ \Rightarrow \text{co}(u_n, x) \] has distribution \( \text{co}(\text{co}(D, D), x) \) over \((-1, x+1)\)

\[ \Pr \left[ u_n \in \left[ x, x+dx \right] \right] = \alpha(x) \, dx \]

\[ \Pr \left[ (u_n, x) \in [u_0(x), u_0(x)+dx] \right] = \alpha(x) \, dx \]

\[ \Pr \left[ Y \in \left[ \text{co}(x, u_0(x)-R \, dx, 0) \right] \right] = \alpha(x) \, dx \]

\[ \Pr \left[ Y \in \left[ y, y+dy \right] \right] = \gamma v(x_0) \left( \frac{1}{2} \cos x \right) \left( \frac{1}{2} \cos x \right) \frac{1}{\sqrt{1-y^2}} \, dy \]

\[ \frac{dy}{dx} = \pm \sqrt{1+y^2} \left( 2y+1 \right) \]

Let \( \frac{dy}{dx} \), similar above, then we compute \( f(y(x)) \) and find

for the integrals of thread over \( n = 100, 1000, 1500 \) iterations

\[ f(y(x)) = \frac{1}{2} \left( 2y - 1 \right) \]

\[ \text{Questions is of course :} \]

\[ \text{What is the effect of keeping the same sign all the rest ?} \]
FIRST CYCLE IN EVOLVING GRAPHS
FIRST CYCLE IN EVOLVING GRAPHS

RANDOM GRAPHS

Returning to original problem:

Evaluate

\[
\int_0^\infty \frac{2}{n^m} \left( 1 - \frac{x}{n} \right)^{\frac{1}{2}} \ln \left( \frac{x}{n} \right) \frac{dx}{x} \]

Very exact (numerical) saddle point with \( n \) & \( \alpha \) exponentially small.

Value of the integrand zeroes at \( n = 1 \) of

\[
\int_0^1 \frac{2}{n^m} \left( 1 - \frac{x}{n} \right)^{\frac{1}{2}} \ln \left( \frac{x}{n} \right) \frac{dx}{x} = \frac{\pi}{2n} \]

\( n = \frac{5}{3} \)

\( n = 10 \)

\( n = 15 \)

\( n = 20 \)

\( n = 30 \)

\( n = 40 \)

\[ 0.31452, 2.9185, 8.0336, 15.929, 36.9734, 66.1212 \]

This is a bit strange!

(been to your house?)

|x|\n|---|---|---|---|---|---|---|---|---|---|---|---|
|15|10|5|n=5|n=10|n=15|n=20|n=30|n=40|n=50|n=60|n=80|n=160|n=320|
|0.3231|0.8736|1.2322|2.0326|3.0126|4.0670|5.0984|6.1680|7.0532|
|x|\n|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
|2.4926|8.1912|9.0532|9.0052|

\( n = 50 \)

| Similar | Similar | Similar | Similar |

Conclusion: For \( n = 50 \), the saddle point approximation is not unreasonable.

From numerical data

\( n = 60 \); the profiles of both functions are quite similar (NB: exact value \( \frac{3}{2} \) very close, eg \( \frac{\pi}{\sqrt{2}} \)).

N.B.: # of discontinuities points should not add up to 0.
FIRST CYCLE IN EVOLVING GRAPHS

The saddle point is:

Equation is:

Saddle points for general $\lambda$

We have finally:

\[
\lambda = \frac{\sqrt{2m n}}{3} \left( 1 - \frac{1}{4} - \frac{10^2}{140} \right) \approx 0.3
\]

\[
\lambda > \frac{n}{4} \quad \text{Solution is} \quad \lambda = \frac{A}{\sqrt{m(n-\frac{1}{2})}} \quad \text{sad}
\]

\[
\lambda < \frac{n}{4} \quad \varepsilon \propto (1-\lambda) \Leftrightarrow \varepsilon \approx 0. \quad \text{So act}\quad y = \lambda (1+y)
\]

Also we find:

\[
-\frac{n \eta + \frac{n \eta}{\lambda (1-\lambda) + 1} - \frac{n \eta}{\lambda (1-\lambda) + 1} + \frac{n \eta}{(1-\lambda) + \frac{\lambda \eta^2}{(1-\lambda) + \frac{\lambda \eta^2}{1-\lambda}}}}{\eta} = 0
\]

\[
\Rightarrow \eta \left( \frac{n-1}{n} + \frac{A}{\lambda} + 5.0 \right) = 0 \quad \Rightarrow \eta \approx \frac{A}{n} \quad \sigma = \lambda \left( \frac{n}{1-n} + 5.0 \right)
\]
FIRST CYCLE IN EVOLVING GRAPHS

\[ e^{2} - e^{3} = 0 \]
\[ k = \frac{1}{\lambda} \]

\[ e = c_{1}t + c_{2}t^{2} + c_{3}t^{3} + \ldots \]
\[ \frac{1}{\sqrt{\lambda k}} \]

\[ e = -\frac{1}{2\lambda k^{3}m} + \left( \frac{5}{8k^{3} - 2k^2} \right) \frac{1}{\sqrt{k^3 m}} \]

\[ \delta = \frac{1 - e}{e} \quad \text{saddle pt.} \]

\[ \lambda = 1 - \frac{1}{\lambda} \]

\[ h(t) = \frac{1}{\lambda} \log(1 - \delta) \]

\[ h(t) = \frac{1}{\lambda} \left( \frac{1}{2} (1 - \delta) \right) \]

\[ \lambda = \frac{\lambda^{2}}{1 - \delta} \]

\[ k^{2} \approx k^{3} \]

\[ e^{2} - e^{3} \Rightarrow k^{2} \gg h^{2} \]

\[ \Delta \approx \frac{1}{2} \lambda^{2} \]

\[ \frac{1}{6} \lambda^{2} (1 - \lambda)^{6} \]

\[ e \approx \frac{1}{2} (1 - \log(1 - \delta)) - \log(1 - \delta) \]
FIRST CYCLE IN EVOLVING GRAPHS

Exact Values: \( n = 5, 10, 15, 20, 25 \)

\[
l(n) = \frac{4}{\ln(2)} \left( 1 - \frac{1}{n} \right) + \frac{\lambda + 2}{\lambda(1-\lambda)^2} \left( 1 + \frac{2(2\lambda + 5\lambda - 1)}{(1-\lambda)^3} \left( 1 - \frac{1}{n} \right) \right)
\]

\[\Rightarrow \text{Everyday is a function of } \frac{4}{\ln(2)} \left( 1 - \frac{1}{n} \right)
\]

- If \( \lambda < 1 \), \( n \)-th is a \( n \)-th power \( n(1-\lambda)^\alpha \). 
  \[\Rightarrow \text{behavior is } \frac{\sqrt{n}}{n^{\alpha}} = O(n^{\alpha}) \]  
  \( \text{except if } \alpha \to 0 \).
- If \( \lambda > 1 \), \( n \)-th is \( n \)-th power \( n(1-\lambda)^\alpha \).
  \[\Rightarrow \text{behavior is } \frac{\sqrt{n}}{n^{\alpha}} = O(1) \]

All this from channel theory.

**WARNING** Final expression is of the form

- \( h_0(\cdot) \)
- \( (\log(\cdot)) \)
- \( \frac{1}{\ln(2)} \)

**Because** one may not expand terms in \( t \) with a factor of \( m^{\frac{1}{2}} \) in the exponential at least.

\[\Rightarrow \text{check more carefully ...}
\]

For instance

\[
\frac{\ln(2)}{2} \log(1 + \frac{2}{n}) = \frac{\ln(2)}{2} \left( 1 + \frac{1}{n} \log(1 + \frac{2}{n}) \right)
\]

\[\Rightarrow \text{But when } \lambda = 1 - \frac{1}{n^{1/3}} \]

\[\Rightarrow \text{An } n \text{-th power } \Rightarrow \text{can't expand in powers of } \frac{1}{n}
\]

Find suitable \( m \) instead of \( n^{1/3} \)

Elegant solution would be

- Expand all in \( \log(1 + t) \)
- Taylor expand
- Substitute for \( n \)-th power

\[\text{93}\]
FIRST CYCLE IN EVOLVING GRAPHS

\[
\lambda > 1 \quad \lambda = \frac{1}{1 - \mu} - \sqrt{3} \quad \text{Scaling is Good!}
\]

Does not work when \( \lambda < 1 \) or \( \mu > 0 \) since the primax integral diverges.

If we have \( \mu = \mu_0 \), then \( \mu = 0 \) should be \( \mu = \mu_0 \).

\[\phi = \phi_0 \quad \text{small}\]

But we get a function \( \phi_0 \).

So take a smaller \( \phi_0 \) and let \( \phi \to 0 \) slowly.

Thus, the contribution to come from a smaller \( \phi_0 \) is \( \frac{1}{1 - \mu} \).

\[
\text{chance = 2} \quad \text{or \ chance = 3}
\]

Equation for saddle point:

\[
\rho = (x^2 - y^2) e^{-\frac{x^2 + y^2}{2}}
\]

Starting again, \( \frac{1}{1 - \mu} \) only occurs around \( \rho = y = x^2 \)

\[
\text{always} 
\]

Then repeat the saddle point (numerical) iterations:

\[
\text{Time is quite clearly \( \psi \) and \( \mu \)}
\]

A quick simulation for walking time (NB: we don't avoid disconnected edges).

\[
\begin{array}{c|c|c|c|c}
\hline
n & \mu & \text{average} & \text{std error} & \text{result} \\
\hline
100 & 4 & 1.01215 & 0.0000 & 1.01215 \\
500 & 5 & 21.4122 & 0.0000 & 21.4122 \\
1000 & 6 & 43.7824 & 0.0000 & 43.7824 \\
2000 & 7 & 84.6988 & 0.0000 & 84.6988 \\
\end{array}
\]
FIRST CYCLE IN EVOLVING GRAPHS

Cycles in a random graph

Assumption: uncorrelated edges, no edge chosen twice, no self loops.

A) Let $C_n(k)$ be the # of configurations satisfying the condition $Q_k$ and let $P_n(Q_k)$ be the probability for $Q_k$ to be satisfied. Then, we have $N=\sum P_n(Q_k)$:

$$P_n(Q_k) = \frac{(k-1)!}{N(N-1)...(N-k)} C_n, k \in \mathbb{Z}^+$$

Some formula works in conditional expectations. Formula (4) gives the relation between the combinatorial model (counting) and the probabilistic model.

The $C_n(k)$ can be computed by the theory of the repeated grazing function. For instance, if $Q_k$ is a cycle has length $e$, then:

$$\sum C_n(k) e^k \frac{1}{m^k} = \frac{1}{2} \frac{y}{(1-y)}$$

where $y = y(\gamma)$ and $y(\gamma) = \gamma e^\gamma$. Hence for instance the expectation of cycle length is

$$\mathbb{E}[\text{cycle length}] = \sum C_n(k) \frac{1}{m^k} \frac{1}{2} \frac{y}{(1-y)}$$

Thus for instance the expectation of cycle length is

$$\mathbb{E}[\text{cycle length}] = \sum C_n(k) \frac{1}{m^k} \frac{1}{2} \frac{y}{(1-y)}$$

(4) Transformation (4) has an integral representation related to the Eulerian Beta integral. Namely:

$$P_n = \int_0^1 C\left(\frac{k}{m}x\right) (1-x)^{k-1} dx$$

where $C(x) = \sum C_n(k) \frac{k^x}{m^x} (\gamma)^x$. No node, $k$ edges.
FIRST CYCLE IN EVOLVING GRAPHS

\[ p_0 = n! n^{-n} \int_0^\infty \frac{z^n}{(1 + z)^N} \left[ \frac{(3 \beta)_{\beta}}{\Gamma(3 \beta)} \right] \frac{dz}{z} \quad (3b) \]

and setting \( n = N \beta = N \beta_0 \):

\[ p_0 = n! n^{-n} \int_0^\infty \frac{z^n}{(1 + z)^N} \left[ \frac{(\beta)_\beta}{\Gamma(3 \beta)} \right] \frac{dz}{z} \quad (3c) \]

The idea is then to evaluate \( C(\beta, \beta_0) \) numerically for fixed \( \lambda \) (actually only \( \lambda = 0 \) matters somewhat)

by

1. The Cauchy integral (of course !)
2. The change of variables of the proof of Lagrange Inversion Theorem
3. Finally saddle point method.

For instance if we take as “parameter” \( \lambda \) the expectation of cycle length \( -3 \), by (C2)

\[ R_n = n! n^{-n} \int_0^\infty \frac{z^n}{(1 + z)^N} \left[ \frac{\beta \lambda}{(\lambda + \beta)} \right] \frac{dz}{z} \quad (3d) \]

and applying (C3) represents some challenge.

1. For fixed \( \lambda \), 2 cases appear

   \[ \lambda < 1, \quad \lambda > 1 \]

   The saddle point \( \sigma = (\lambda, n) \) satisfies an algebraic equation of degree 3, but it needs to be picked carefully.

   For \( \lambda < 1 \)
   \[ \sigma = \lambda + \sigma(\lambda) \]
   \[ 1 > 1 \]
   \[ \sigma = \lambda - \sigma(\lambda) \]

   and using asymptotic manipulation systems, I find with \( I_n(\lambda) \)

   \[ \lambda < 1 : \quad I_n(\lambda) \sim \frac{1}{n^{1/2}} e^{\frac{n^{1/2}}{\lambda}} = 0(n) \text{ for fixed } n \]
   \[ \lambda > 1 : \quad I_n(\lambda) \sim e^{\frac{n}{\lambda}} e^{\frac{n^{1/2}}{\lambda}} = 0(n) \text{ for fixed } n \]

   Thus, the contribution should be localized around \( \lambda = 1 \).

   At \( \lambda = 1 \):

   \[ I_n(1) \sim \frac{n^{1/2}}{2} e^{-1/2} = 0(n) \]

   Thus the problem is to find how \( \mu(\lambda) \) behave for \( \lambda \) very close to 1 but a function of \( n \) itself.

   2. It takes a little work to find that the proper scaling factor is

   \[ \mu = n^{-1/2} \]

   and we consider now

   the saddle point \( \lambda = 1 - \frac{1}{\epsilon} = 1 - n^{-1/2} \) is fixed.

   Then the saddle point \( \sigma = (\lambda, n) \) has an asymptotic expansion for \( n \to \infty \):

   \[ \sigma = \lambda - \xi(\mu) + \frac{\mu + 1 + \mu \xi(\mu)}{2(1 + \mu \xi(\mu))} n^{1/2} + O(\mu^2) \]

   (more terms are necessary and have been computed with maple).

   In (6) \( \xi(\mu) \) is an algebraic function of degree 3 defined by
FIRST CYCLE IN EVOLVING GRAPHS

\[ d_1 \phi^2 - d_0 \phi + 1 = 0 \]  

As \( n \to +\infty \), \( \phi \) varies continuously from 0 to +\( \infty \) as \( \nu \) goes from -\( \infty \) to +\( \infty \).

Using the saddle point method with \( e^{-p} \) exposed as functions of \( \mu \) and \( \phi = \phi(\mu) \) (only implicitly defined), I found at some labour that:

\[ I_n(\lambda) \approx \frac{\sqrt{n}}{2} e^{\frac{\pi}{2}} \frac{e^{\frac{\mu}{2}} e^{\frac{\pi^2}{2}}} {\sqrt{2 + \pi^2}} \]  

Thus after a rescaling one has \( \lambda = n^{-\frac{1}{2}} \mu \) from 8:

\[ K_n \sim e^{\frac{\mu^2}{2}} n^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{\frac{\pi^2}{2}} \frac{e^{\frac{\pi^2}{2}}} {\sqrt{2 + \pi^2}} d\mu \]  

which can be re-expressed using \( \mu \) as an independent variable.

Theorem:

\[ K_n \sim e^{\frac{\mu^2}{2}} n^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{\frac{\pi^2}{2}} \frac{e^{\frac{\pi^2}{2}}} {\sqrt{2 + \pi^2}} d\mu \]  

which is then empirically proportional to \( n^{\frac{1}{2}} \).

I have also written simulation programs. For instance, I have found that based on:

- 100 simulations \( n = 800 \) \( \Rightarrow K_{100} \approx 6.75 \)
- 100 simulations \( n = 6400 \) \( \Rightarrow K_{6400} \approx 13.25 \)

Not too bad!

Other applications: study probability distribution of cycle-length, size of cycles, connected, waiting times, other models, replicated edges....
FIRST CYCLE IN EVOLVING GRAPHS
FIRST CYCLE IN EVOLVING GRAPHS
FIRST CYCLE IN EVOLVING GRAPHS

26/11

\[ 2(y) = (x - y)(x + y) \]

\[ 2(x) = 2 - y \]

\[ 2 \frac{1}{2} + \frac{1}{2} \log 2 = \log 0 + \frac{1}{2} \log 0 \]

\[ X = (-x) + \frac{1}{2} + \frac{1}{2} \log 2 \]

\[ Y = (-y) + \frac{1}{2} + \frac{1}{2} \log 2 \]

\[ \frac{1}{2} \sqrt{2} = \sqrt{\log 2} \]

\[ \sqrt{2} = \sqrt{\log 2} \]

\[ \text{Checked numerically} \]

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2/10/85

\[ \text{SUMMARY of THINGS TO BE DONE} \]

\[ \text{HAVE APPEARED SINCE 19/08 Jan 85} \]

1. BIT ’85
   \[ \text{Appropriate Publishing} \]
   \[ \text{PC: P. Ch. Venkatesh} \]
   \[ \text{IEEE IPCS} \]
   \[ \text{PC: M. Kanakasabapathy} \]
   \[ \text{ACM} \]
   \[ \text{PC: J. B. Venkatesh} \]
   \[ \text{PC: R. Venkatesh} \]

2. TCS
   \[ \text{PC: M. Kanakasabapathy} \]
   \[ \text{PC: J. B. Venkatesh} \]

3. TACO
   \[ \text{PC: M. Kanakasabapathy} \]
   \[ \text{PC: J. B. Venkatesh} \]

4. STACS
   \[ \text{PC: S. S. Venkatesh} \]
   \[ \text{PC: J. B. Venkatesh} \]

5. FSTTCS
   \[ \text{PC: S. S. Venkatesh} \]
   \[ \text{PC: J. B. Venkatesh} \]

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29 September 85

\[ \text{Probabilistic Graphs} \]

\[ \text{Author: A. K. Singh} \]

\[ \text{Title: Graphs in Evolutionary Systems} \]

\[ \text{Abstract:} \]

\[ \text{From Pickardt and Albert's paper in "Information et Intelligence"} \]

\[ \text{Let } H(p) \text{ be the Pome-Down sequence.} \]

\[ \text{Proof:} \]

\[ \text{If } H(n) = H(n-1) + H(n-2) + \cdots + H(1) \]

\[ \text{Then:} \]

\[ \sqrt{2} = \frac{1}{2} \left( \frac{3}{2} \right)^{n} \]

\[ \text{My main content from Pickardt and Albert's paper is that the sequence:} \]

\[ \text{is not ready to be tried} \]

\[ \text{Proof:} \]

\[ \text{If } H(n) = H(n-1) + H(n-2) + \cdots + H(1) \]

\[ \text{Then:} \]

\[ \sqrt{2} = \frac{1}{2} \left( \frac{3}{2} \right)^{n} \]

\[ \text{checked numerically} \]
SYNERGISTIC INTERACTION

Mathematics

Computers  Communications
BACKHOUSE’S CONSTANT

Radius of cv of $\frac{1}{1 + \sum_{k \geq 1} p_k z^k}$

A030018 Coefficients in $1/(1+P(x))$, where $P(x)$ is the generating function of the primes.
1, -2, 1, -1, 2, -3, 7, -10, 13, -21, 26, -33, 53, -80, 127, -193, 254, -355, 527, -764, 1149, -1699, 2436, 3563, 5133, -7352, 10819, -15863, 23162, -33887, 48969, -70936, 103571, -150715, 219844, -320973, 466641, -679232, 988627, -1437185, 2094446, -3052743 (list: graph; refs; listen; history; text; internal format)
OFFSET 0,2
COMMENTS a(n+1)/a(n) => -1.4560749485826896714. - Zak Seidov, Oct 01 2011.

Backhouse's constant

From Wikipedia, the free encyclopedia

Backhouse's constant is a mathematical constant founded by N. Backhouse and is approximately 1.456 074 948.

It is defined by using the power series such that the coefficients of successive terms are the prime numbers:

$$P(x) = 1 + \sum_{k=1}^{\infty} p_k x^k = 1 + 2x + 3x^2 + 5x^3 + 7x^4 + \cdots$$

and where

$$Q(x) = \frac{1}{P(x)} = \sum_{k=0}^{\infty} q_k x^k.$$ 

Then:

$$\lim_{k \to \infty} \left| \frac{q_{k+1}}{q_k} \right| = 1.45607 \ldots \text{ (sequence A072508 in OEIS).}$$

The limit was conjectured to exist by Backhouse which was later proved by P. Flajolet.
Backhouse's Constant

Let $P(x)$ be the formal power series whose $n$th term has coefficient equal to the $n$th prime number:

$$P(x) = \sum_{k=0}^{\infty} p_k x^k = 1 + 2x + 3x^2 + 5x^3 + 7x^4 + 11x^5 + 13x^6 + \ldots$$

Let $Q(x)$ be the formal power series defined by

$$Q(x) = \frac{1}{P(x)}$$

Thus $Q(x)$ is the formal reciprocal of $P(x)$ as a power series. Observe that this is pure formal algebra; no questions of analytical convergence are involved at all.

$Q(x)$ is an alternating series whose coefficients $q_n$ are monotonically increasing in magnitude. Nigel Backhouse has observed that the ratios of successive coefficients tend to a certain constant, i.e., it appears that

$$\lim_{n \to \infty} \frac{q_{n+1}}{q_n} = 1.4560749485826896713995953511654356.$$ 

In a personal communication, Backhouse wrote:

The approximation given was generated in 37 seconds using Maple V (release 3) in batch mode on a Silicon Graphics Irix 4D. $P(x)$ was taken to 550 terms and $Q(x)$ produced as the Taylor series of $P(x)^{-1}$.

Unfortunately, I have no references to this result or anything like it. In particular, I have no evidence as to the originality of my observation. I was just curious, as someone with an amateur interest in number theory!

I should, of course, be very interested to hear, if, as a result of your enterprise, someone has anything to add to my rather thin story.

The 35-place decimal approximation above also appears at the CECM Inverse Symbolic Calculator web site. I am grateful to Simon Plouffe for pointing out to me the existence of this constant and to Nigel Backhouse for providing the information on which this essay is based.

Relevant Mathcad files will be included as time permits.
ON THE EXISTENCE AND THE COMPUTATION OF BACKHOUSE'S CONSTANT

Philippe Flajolet, Algorithms Project, INRIA
November 25, 1995
<Philippe.Flajolet@inria.fr>

I. THE PROBLEM

Let \( p(n) \) be the \( n \)-th prime, with \( p(1)=2 \), and define

\[
\begin{align*}
\infty & \\
\cdots & \\
1 & \\
0 & \\
\end{align*}
\]

\[
Q(z) = \frac{1}{P(z)},
\]

where \( P(z) = \prod_{n=1}^{\infty} (1-p(n)z) \).

Nigel Backhouse examines the coefficients \( q(n) \) in the series \( Q(z) = \sum_{n=0}^{\infty} q(n)z^n \).

He notices empirically that the \( q(n) \) alternate in sign and that the ratio between successive values tends a constant equal (up to sign) to \( 1.45607... \) and called now "Backhouse's constant". See the description in Steven Finch's pages on Constant on the web.

II. ANALYSIS

Here is what goes on. By the Prime Number Theorem, we have \( p(n) \sim n \log(n) \), and at any rate \( p(n) \) has a unique zero at \(-2.5486 \pm \) inside the unit disk. Since \( P(0)=1, Q(0) \) is analytic at 0. Thus, by Cauchy's coefficient formula,

\[
\frac{1}{(2\pi i)^n} \oint \frac{Q(z)}{z^{n+1}} \, dz,
\]

where the integration contour is a sufficiently small circle around 0. We observe that \( P(z) \) has a unique zero at \(-2.5486 \pm \) inside the disk of radius 0.75. Thus, integrating along \( |z|=0.75 \) and taking into account the residue of \( Q(z) \) at \( z=0 \) gives us

\[
q(n) = \frac{(-1)^n}{n} (n+1)^{1/2} - 0.75^{n+1}.
\]

where \( a_0 = 1.45607... \) is Backhouse's constant. This formula is quite good as its error term is of order \( 1/0.75^{1.33} \), hence exponentially smaller than the dominant term.

It is possible to go farther by fishing for the next poles. In this way one can find better and better asymptotic expansions of the type

\[
\]

(with suitable modifications if multiple poles were to be encountered), where \( a(0)=\sqrt{a(2)}(1+\cdots) \) and \( a(2) \) is the 1-st zero of \( P(z) \).

There doesn't seem to be real poles apart from 0.

III. A GENERAL REMARK

What we just did is an instance of a general process well known in the analysis of coefficients of meromorphic functions. It is related to methods for coefficient asymptotics, like Darboux's method or singularity analysis, that are especially useful in "analytic combinatorics". An example that is close and that I like to use in teaching coefficient asymptotics is the following.

A composition of an integer \( n \) is a sequence of integers \( >0 \) that sum to \( n \). The number of compositions of \( n \) is \( 2^{n-1} \). How many are there?

Answer: about \( 0.363552631 \times 10^7 \).

Proof: Work with the series \( S(z)=\sum_{n=1}^{\infty} a(n)z^n \) where \( a(n) \approx z^{2^n-3^2+2^n-2^n} \).

Philosophy: This discussion shows how to improve over powers of 2, 3, 5, 7, 11... by a factor of \( 2^n \).

How many are there?

Answer: about \( 0.363552631 \times 10^7 \).

Proof: Work with the series \( S(z)=\sum_{n=1}^{\infty} a(n)z^n \) where \( a(n) \approx z^{2^n-3^2+2^n-2^n} \).

IV. NUMERICAL VALUE OF BACKHOUSE'S CONSTANT

Here is in connection with Simon Plouffe's dictionary of real numbers

http://www.cecm.sfu.ca/projects/ISC.html

the value of Backhouse's constant to some 1300 Digits

(determined in 4 minutes of CPU time with a Maple V.3 program on a DEC Alpha 3000 station.)

\[
1.456074984582686719959331154357653175783747481315402707024
374903905462583999595945361428603919925214363661319486677
51649132122314292035770128374053950749980254652030705808528
\]
V. THE MAPLE PROGRAM

It is just Newton's method applied to $P(z)$, with a close enough starting value, increasing the number of terms in the truncation of $P(z)$ as we proceed. The call to $b(7)$ gives 1357 exact digits in 4 minutes of CPU time.

Digits:=16; # must be there because of Maple's idiosyncrasy
b:=proc(m) # compute more than $10^{12}$ digits of Backhouse's Constant
local ord, x, i, j, P, DP, t;
option remember;
ithprime:=proc(n) option remember; ithprime(n); end;
Digits:=15:
x:=2/5, 45674985686976;
ord:=72;
for i from 1 to m do
Digits:=20+Digits; ord:=2*ord;
P:=1; DP:=0; x[j]:=x;
for j from 1 to ord do
i:=ithprime(j); x[j]:=x[j];
ord:=1+i; t:=ithprime(i); x[j]:=x[j];

od;
x:=x/DP:

od;

RETURN(-1/x):
end;
b(6); # gives 678 exact digits in 70 seconds
b(7); # gives 1357 exact digits in < 4 minutes

```
GRAY CODE FUNCTION

\[
\sum_{0 \leq k \leq n} \binom{n}{k} (-1)^{n-k} g_k, \quad g_n = 2g\left[\frac{n}{2}\right] + \frac{1-(-1)^{\left[\frac{n}{2}\right]}}{2}
\]
GRAY CODE FUNCTION: $g_n \& g_{n-1} - g_n$
GRAY CODE FUNCTION

\[ a_n := \sum_{0 \leq k \leq n} \binom{n}{k} (-1)^{n-k} g(k) \]

\[ \frac{a_n}{(-2)^n} = [z^n] G \left( -\frac{z}{2-z} \right) \quad G(z) := \sum_{j \geq 0} \frac{2^j z^{2^j}}{1 + z^{j+1}} \]

\( n = 43 \) is the first exception that \( a_n/(-2)^n > 0 \)
GRAY CODE FUNCTION

\[ a_n := \sum_{0 \leq k \leq n} \binom{n}{k} (-1)^{n-k} g(k) \]

\[
\frac{a_n}{(-2)^n} = -\frac{1}{2\sqrt{2\pi n}} \int_{-\infty}^{\infty} e^{-\left(\frac{v-n}{\sqrt{2n}}\right)^2} \sum_{3 \leq k \leq L_n + 2} \frac{\sin\left(\frac{1}{2} v \pi\right)}{\cos\left(\frac{\pi v}{2k}\right)} \, dv
\]

Asymptotics of \( \frac{a_n}{(-2)^n \sqrt{n}} \) remains open
OPEN PROBLEMS IN PF’S OEUVRES
[PF44] (P762) & [PF98] (P217): ... ce qui donne lieu à la plus célèbre conjecture de l’informatique

\[ P \neq NP \]

[PF197]:

\[
\sum_{2 \leq k \leq n} \binom{n}{k} \frac{(-1)^k}{\zeta(k)} = O\left(n^{\frac{1}{2}+\varepsilon}\right)
\]

≡ Riemann Hypothesis

RH is also connected to algorithm complexity (with Vallée & Clement): [PF144] [PF157] [PF161]
[PF51] [PF69] [PF72] [PF108]: Height & diameter of BSTs (Devroye, Reed, Drmota, …)

[PF122]: height of quadtrees (Devroye)

[PF112] [PF152]: Quicksort limit law (Fill, Janson, Devroye, Neininger, …)

[PF147] Max deg in planar triangulations (Gao, Wormald)

[PF200] \((1 - \lambda)^{-\frac{1}{3}}\) realizable by stochastic context-free grammar? (Banderier, Drmota)

…
Beginning around 1994, under a series of papers stating a set of conjectures seemingly proven in 2013 by Alkauskas.

Philippe was interested in computing the spectrum:
- for $s = 1$: Euclid algorithm
- for $s = 2$: Gauss reduction algorithm

\[
G_s[f](x) := \sum_{m \geq 1} \frac{1}{(m + x)^{2s}} f \left( \frac{1}{m + x} \right)
\]
For the Gauss-Kusmin-Wirsing operator $G := G_1$

- All eigenvalues $|\lambda_n|$ are **simple** & **strictly**
- They **alternate** in sign: $(-1)^n\lambda_n > 0$
- \[ \lim_{n \to \infty} \frac{\lambda_n}{\lambda_{n+1}} = -\phi^2 \quad \& \quad \lambda_n \sim (-1)^{n+1}\phi^{-2n} \]

Alkauskas announced in 2013 a proof of the conjectures

**arXiv 1210.4083**: “*In this work we prove an asymptotic formula for the eigenvalues of L. This settles, in a stronger form, the conjectures of D. Mayer and G. Roepstorff (1988), A. J. MacLeod (1993), Ph. Flajolet and B. Vallée (1995), ...*”

He asked Brigitte if other experiments were performed. Then with Julien, more computations made by Philippe were found ...
- [PF207]: non-holonomicity of $\cos \sqrt{n}$, $\cosh \sqrt{n}$
- [PF204]: *three-sided prudent polygons*, $g_4 = 1 + \log_2 3$?
- [PF200]: Buffon machine for Euler’s $\gamma$?
- [PF191]: *hidden word statistics, convergence rate to normality*?
- [pF185]: Graeffe polynomials computable at a lower cost?
- [PF174]: *motif statistics under more general models*?
- [PF172]: robustness of interconnection in random graphs, finer properties like variance?
STILL OPEN?

- [PF161]: trie statistics under general dynamic sources
- [PF157]: continued fractions & comparison algorithms, many questions
- [PF144]: continued fraction algorithms, uniformity of quasi-power for MGF?
- [PF69]: linear worst-case time for tree-matching algorithms?
- [PF64]: ambiguity of context free languages, many questions
- [PF54]: collision resolution algorithms in random access systems, limit of stability?
WHAT TO DO WITH THE CAHIERS?

Cahiers ➔ Digital Forms

Web Accessible?

PolyPF Project?