# On the profil-distance of random rooted plane graphs

Olivier Bernardi, **Gwendal Collet**, Éric Fusy

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# Introduction

Study of the **shape** of planar maps:

$\rightarrow$ Chassaing–Schaeffer (2004):
typical distance in rooted plane quadrangulations
with $n$ faces $\sim n^{1/4}$ when $n \rightarrow \infty$
$+$ distance profile converges towards **ISE**

$\rightarrow$ Le Gall (2007, 2012), Miermont (2012):
quadrangulations converge towards the **Brownian map**
when rescaled by $n^{-1/4}$
extended to $2p$-angulations and triangulations

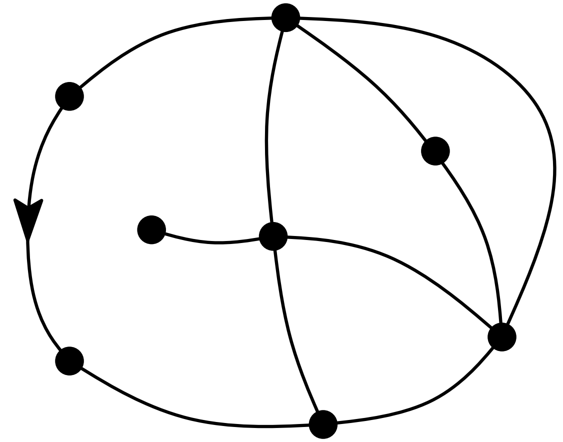
$\rightarrow$ Addario-Berry–Albenque (2013):
simple triangulations and simple quadrangulations
converge towards the **Brownian map**

universal limit?

# Definitions

**Rooted plane graph:**

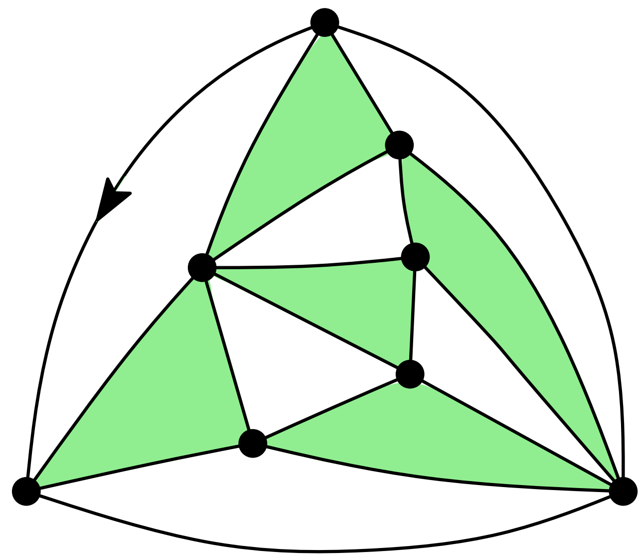
- no loops
- no multiple edges
- $M(z)$ generating series according edges



**Rooted eulerian triangulation:**

- vertices have even degree
- green and white triangles
- $T(z)$ generating series according green triangles

$$T(z) = \sum_{n \geq 1} \frac{3}{n+2} 2^{n-1} Cat_n z^n$$

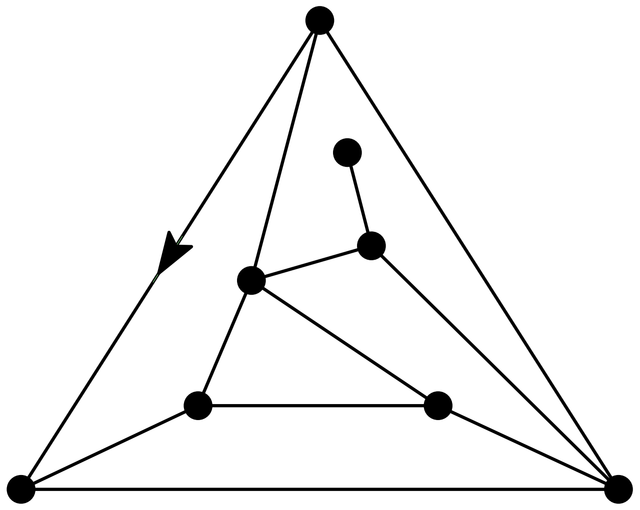


[M. Noy] $M(z) = \frac{z(1+T(z)/z^2)}{1-z(1+T(z)/z^2)}$

# Focus on outertriangular plane graphs

**Outertriangular**
**rooted plane graph**

$C(z)$ generating series according edges



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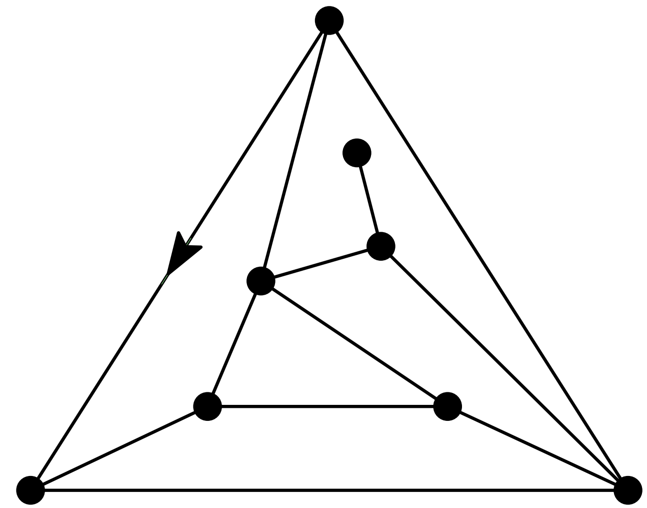

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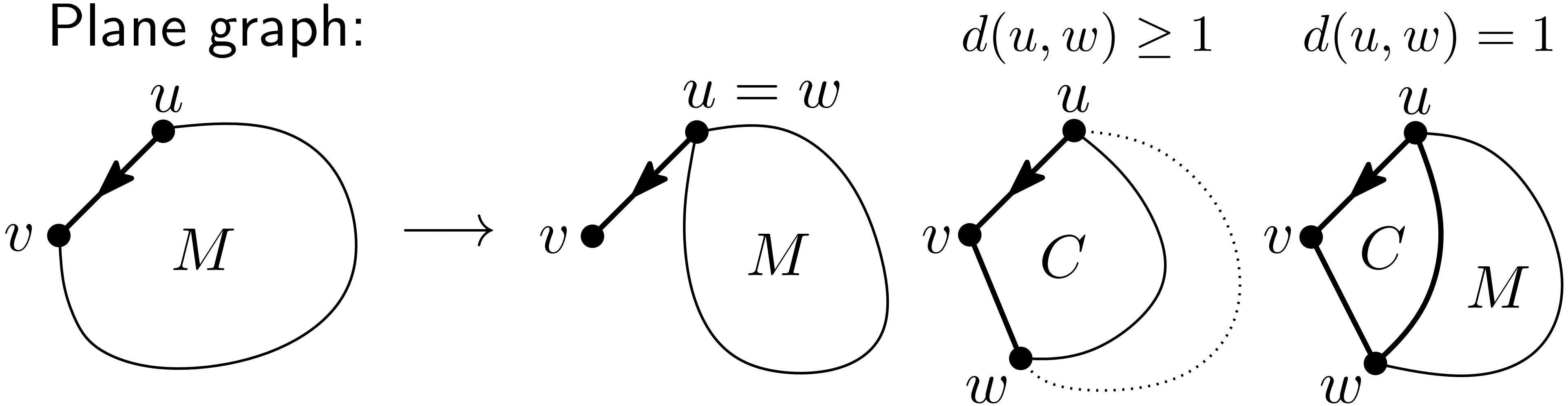
Plane graph:



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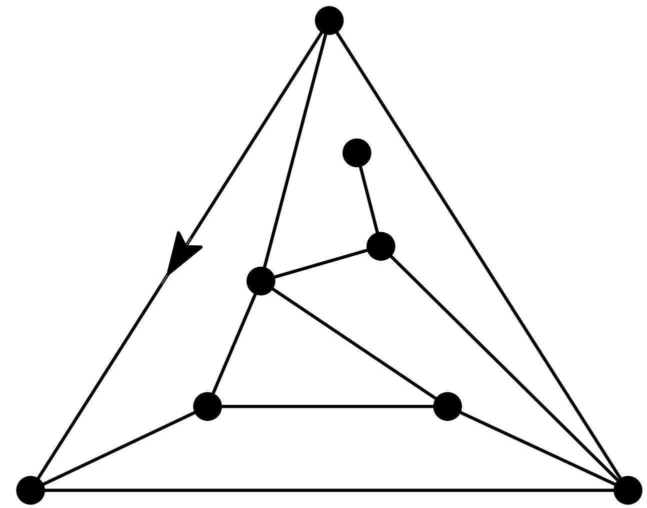

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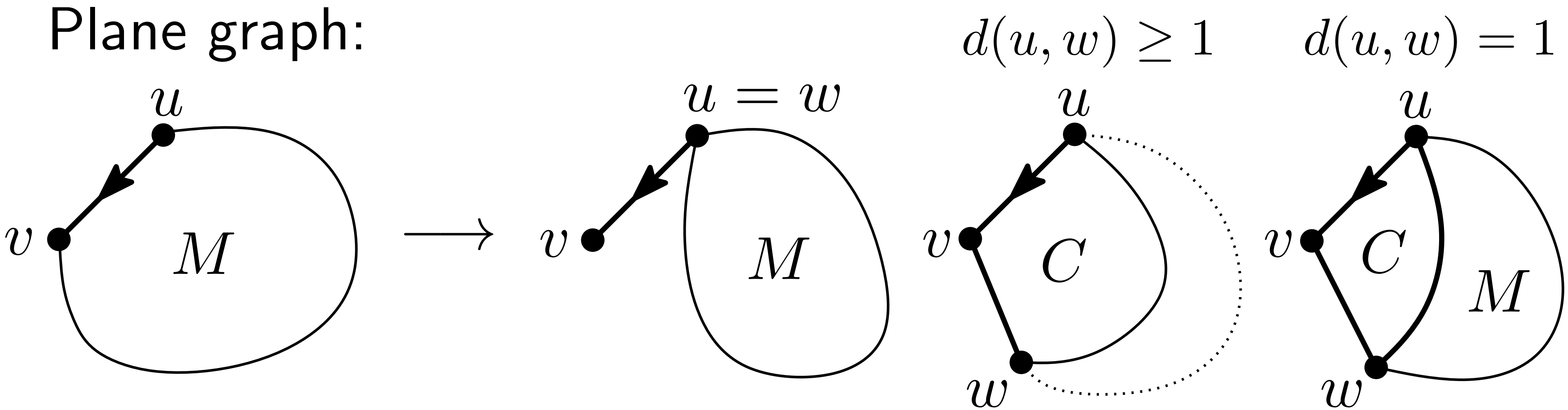
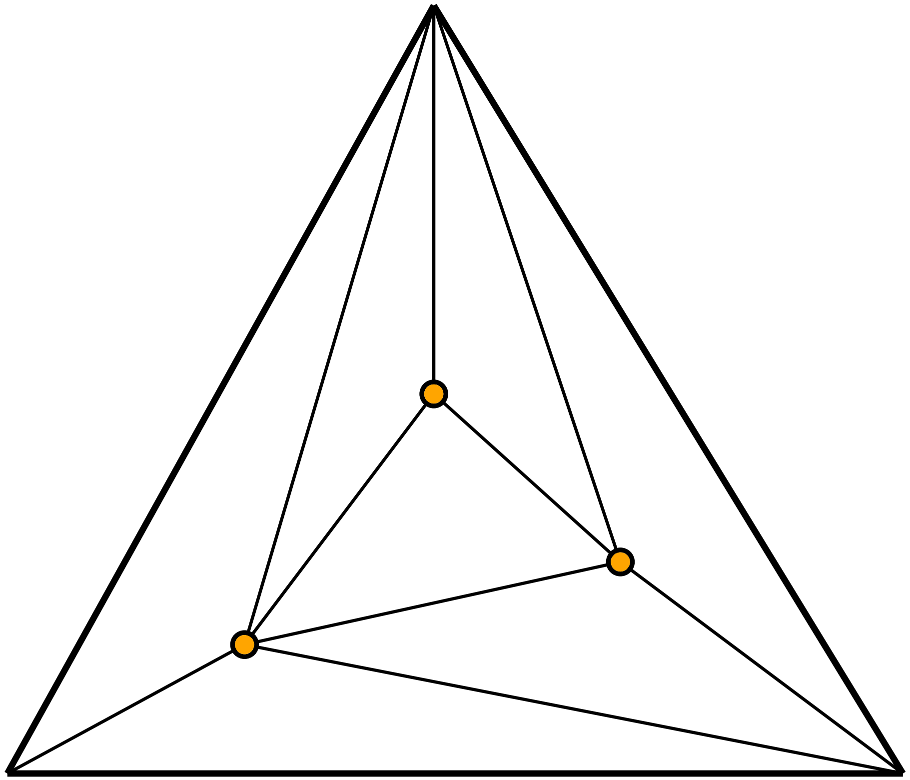



$$\Rightarrow M(z) = z(1 + M(z)) + \frac{C(z)}{z} + \frac{C(z) \cdot M(z)}{z}$$

$$\Rightarrow M(z) = \frac{z(1+C(z)/z^2)}{1-z(1+C(z)/z^2)}$$

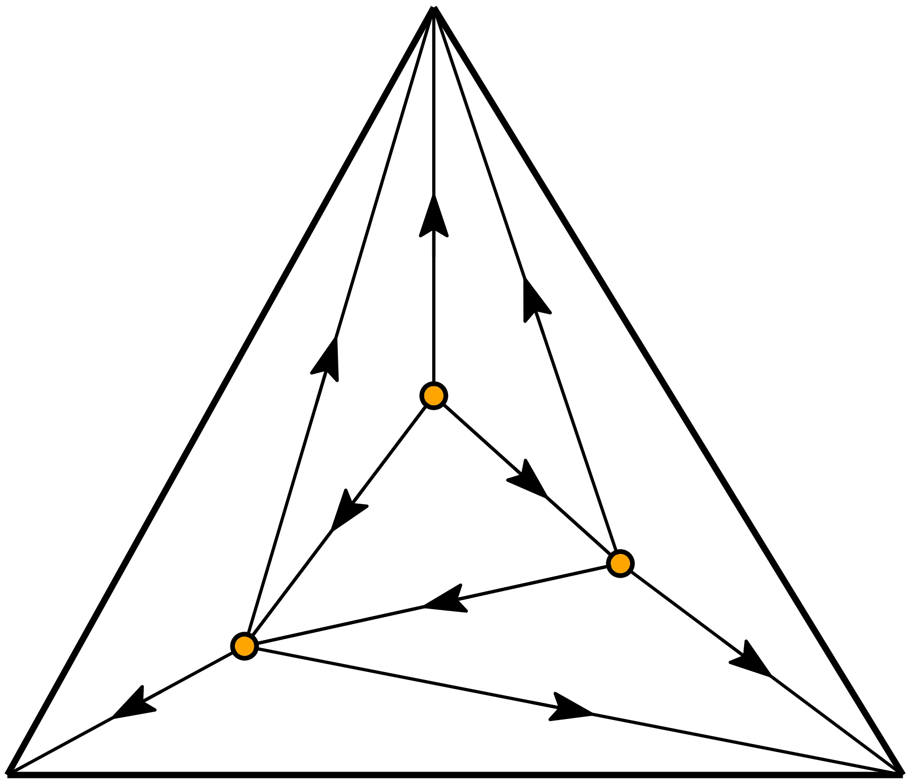
← suggest bijection between outertriangular plane graphs and eulerian triangulations

# From maximal plane graphs to Eulerian triangulations



# From maximal plane graphs to Eulerian triangulations

[Schnyder'89, Brehm'02]



*Canonical orientation:*

3 outgoing edges at vertices
outer-accessibility
no clockwise circuit

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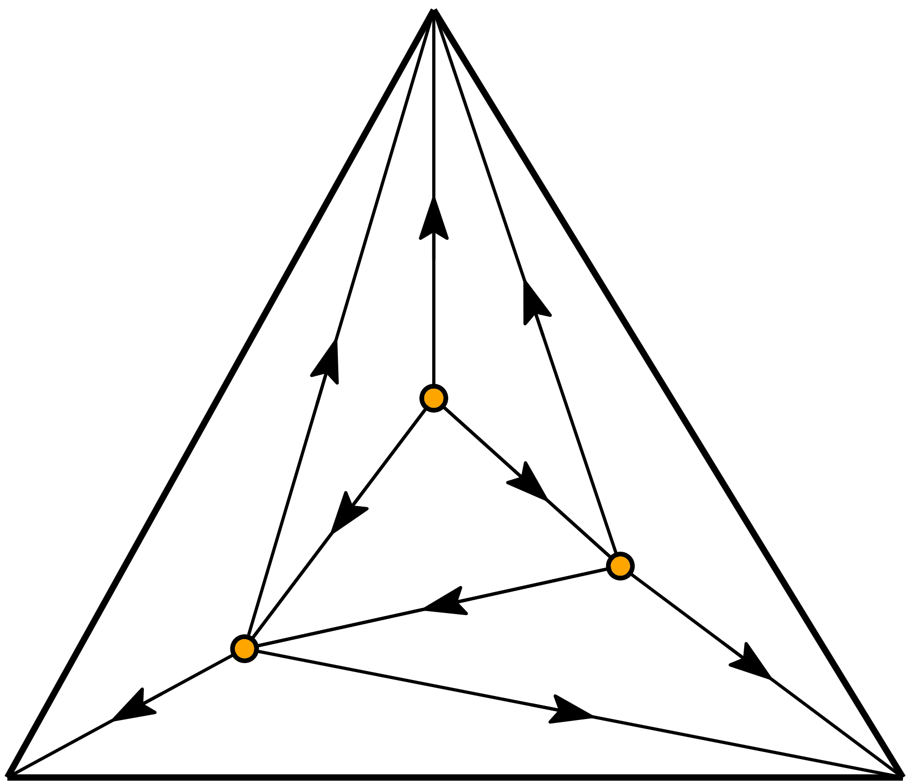

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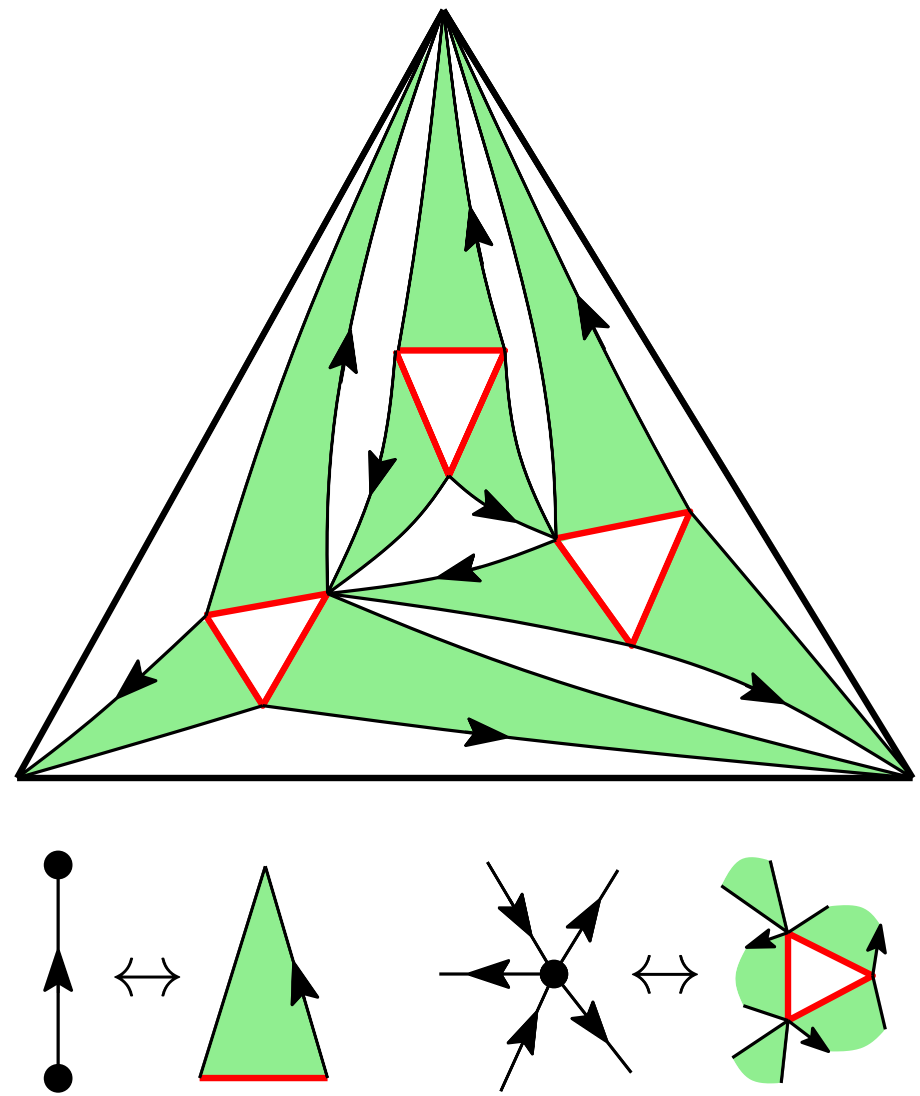

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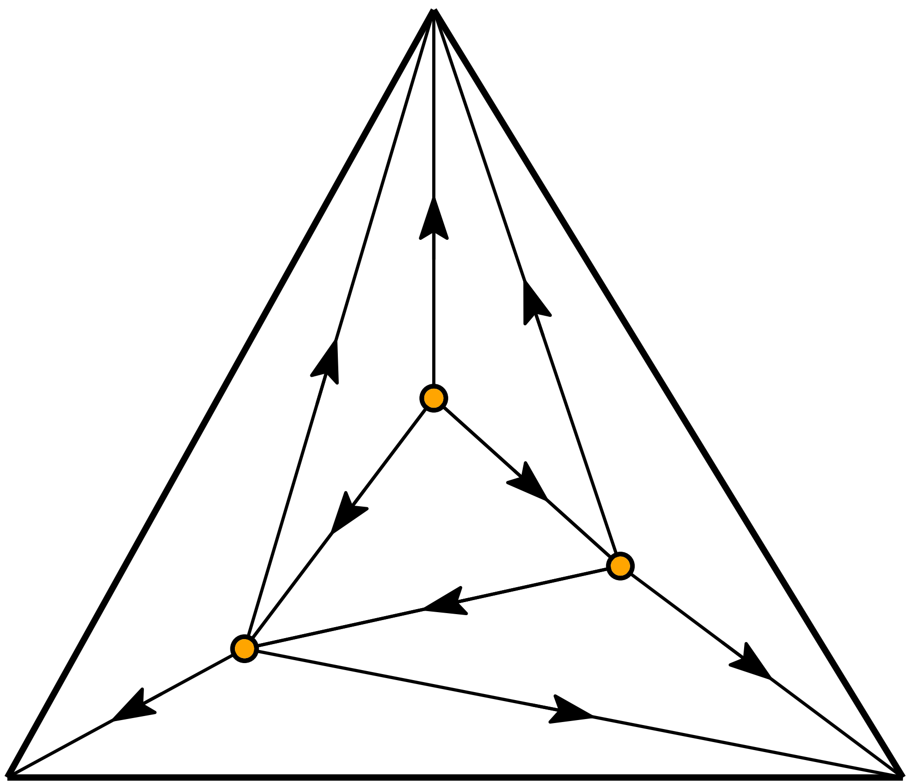

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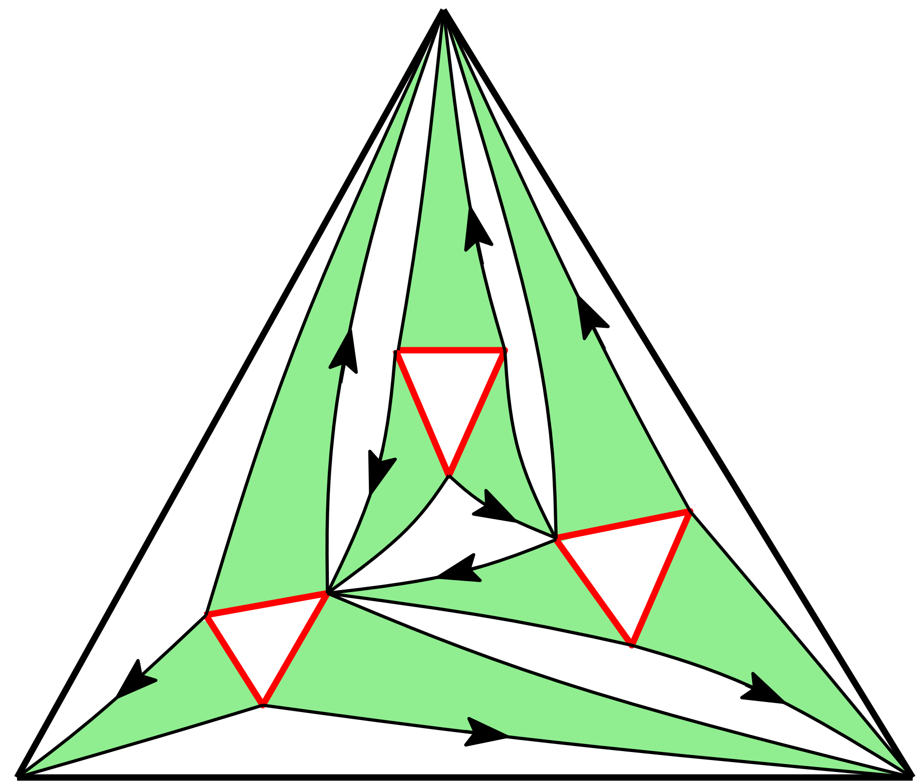
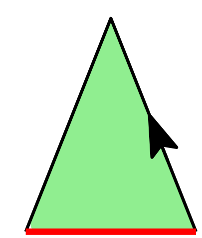


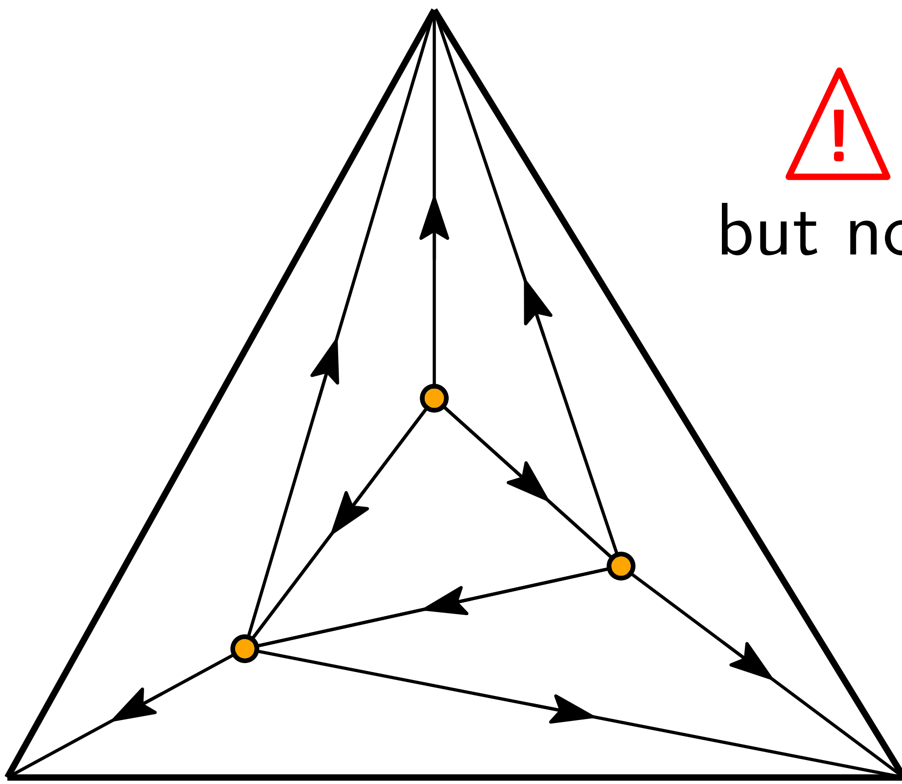
*Canonical orientation:*

1 outgoing edge at vertices
oriented forest
with $3$ components
on outer vertices



# From maximal plane graphs to Eulerian triangulations

⚠ Injective
but not surjective



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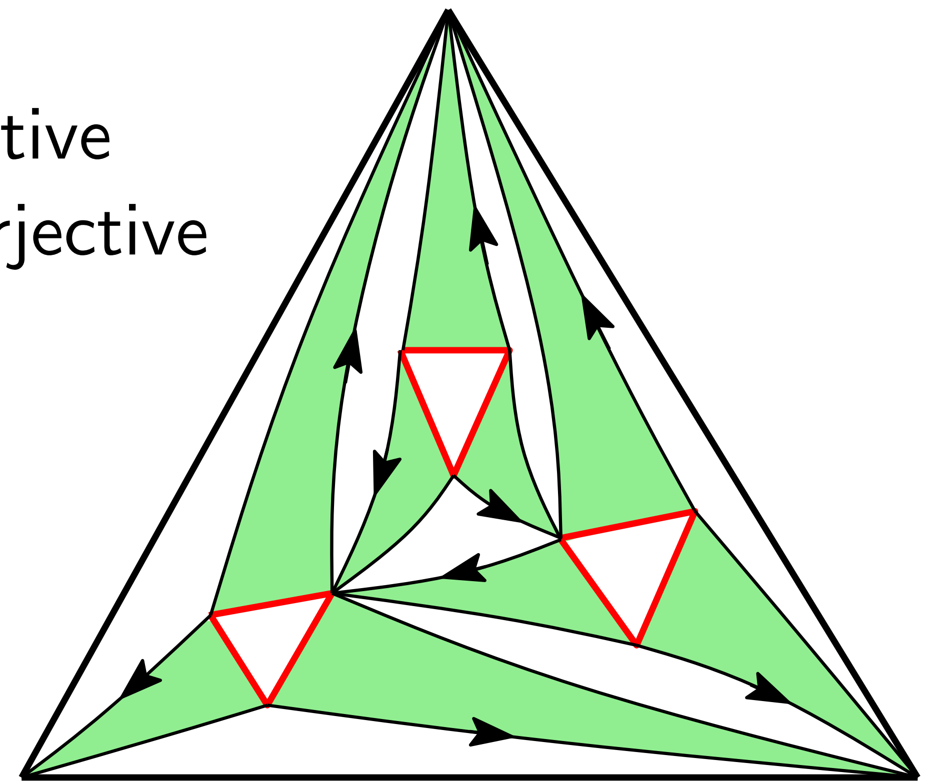

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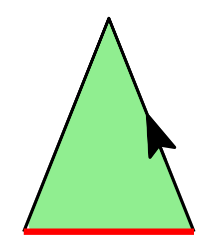
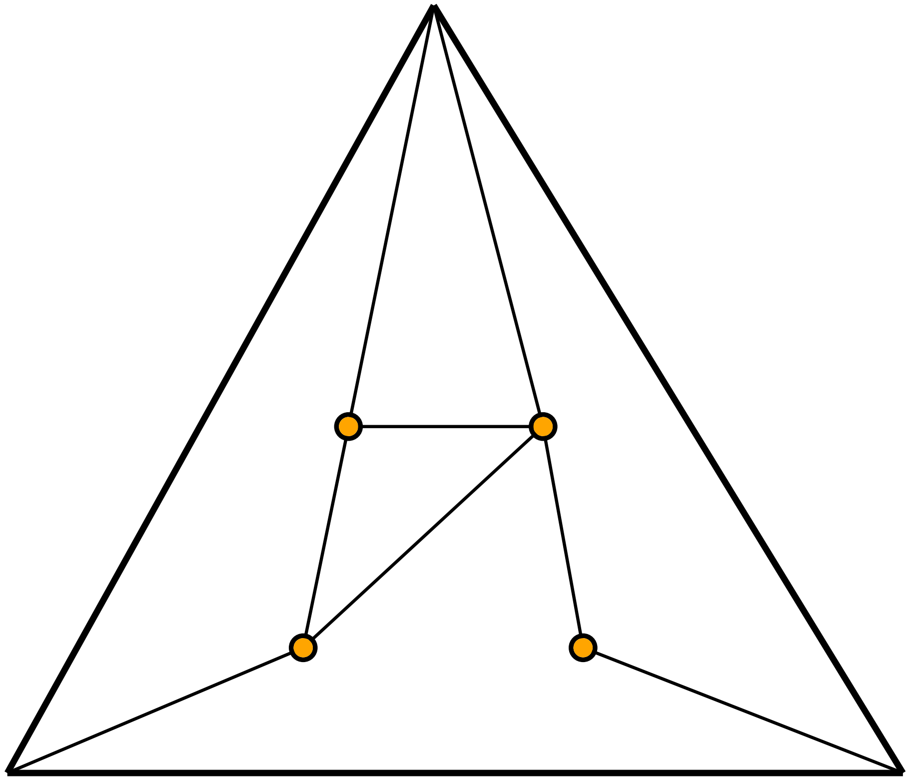


# Bijection for outertriangular plane graphs



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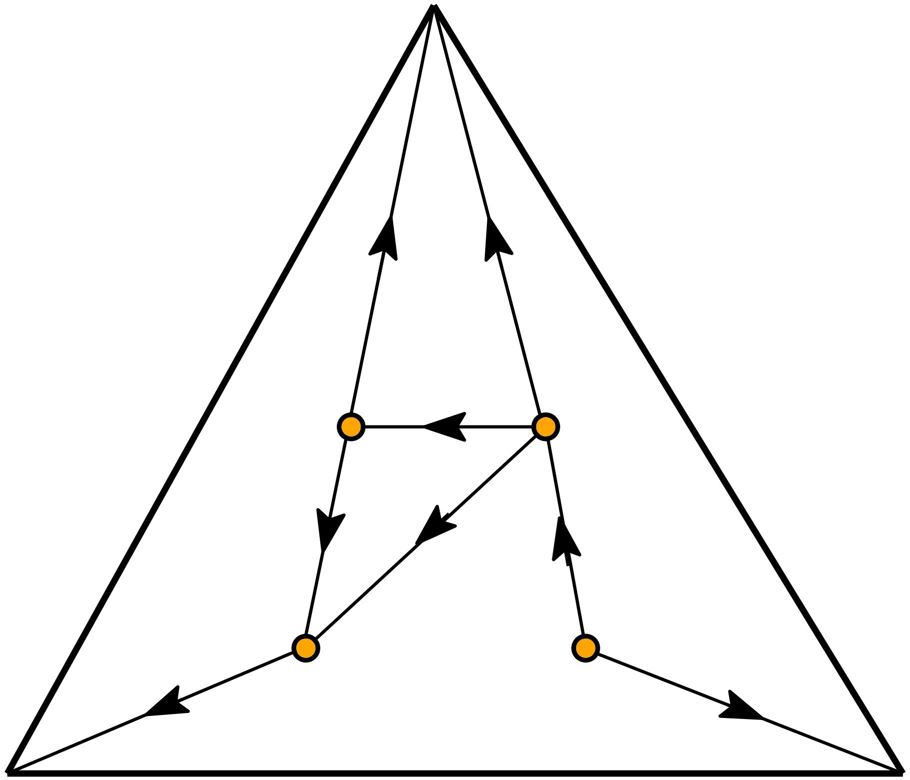

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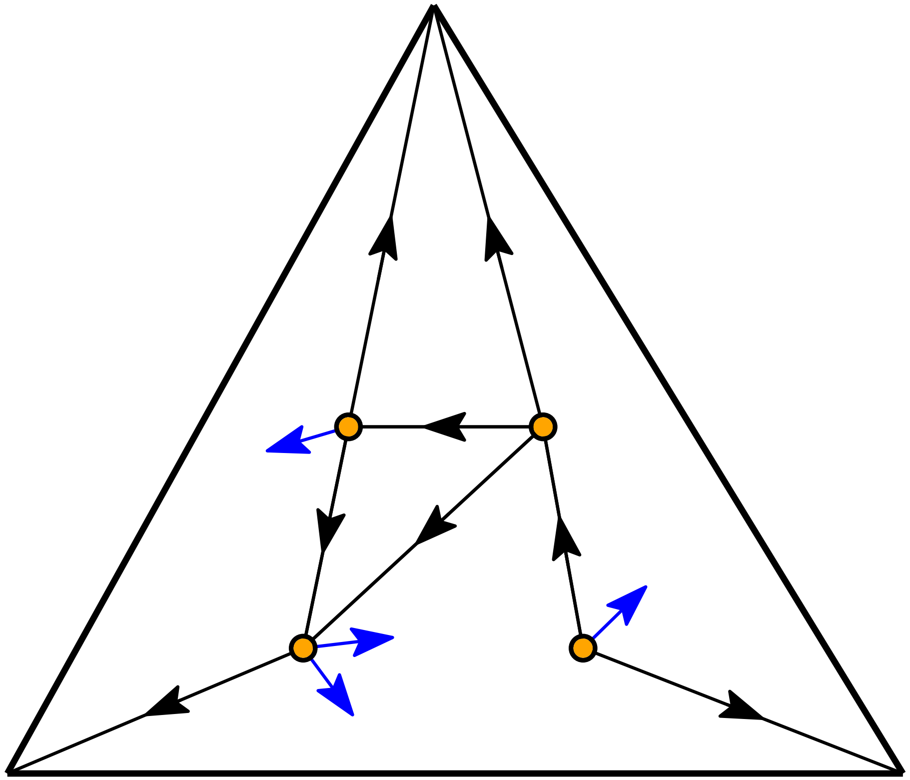

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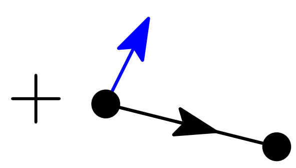
face of degree $d+3$ : $d$ arrows



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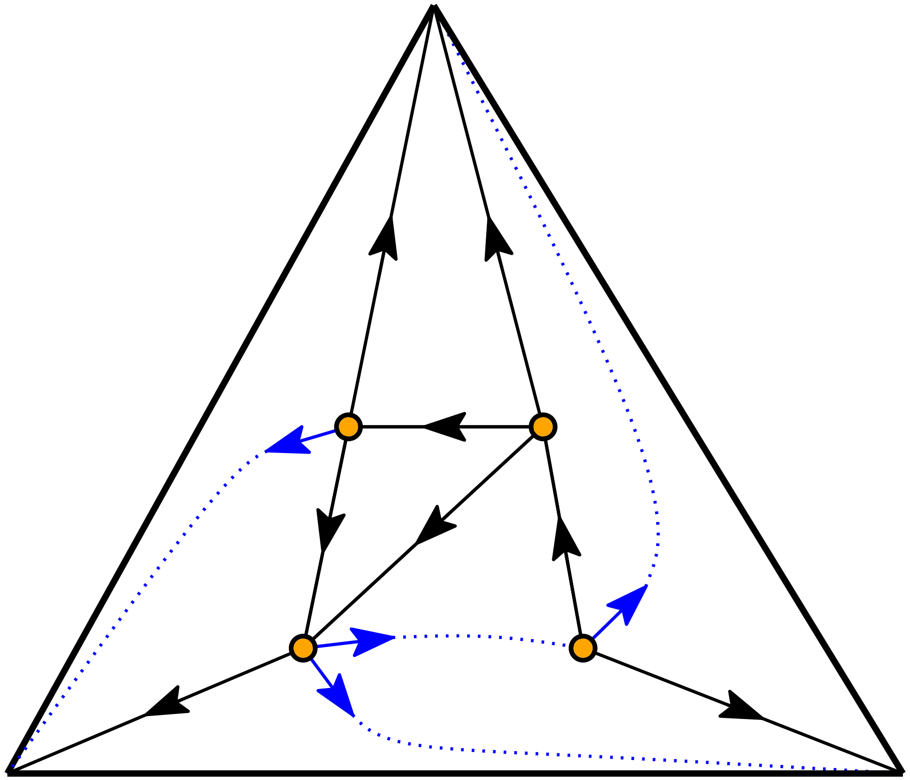

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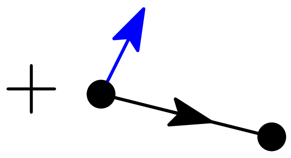

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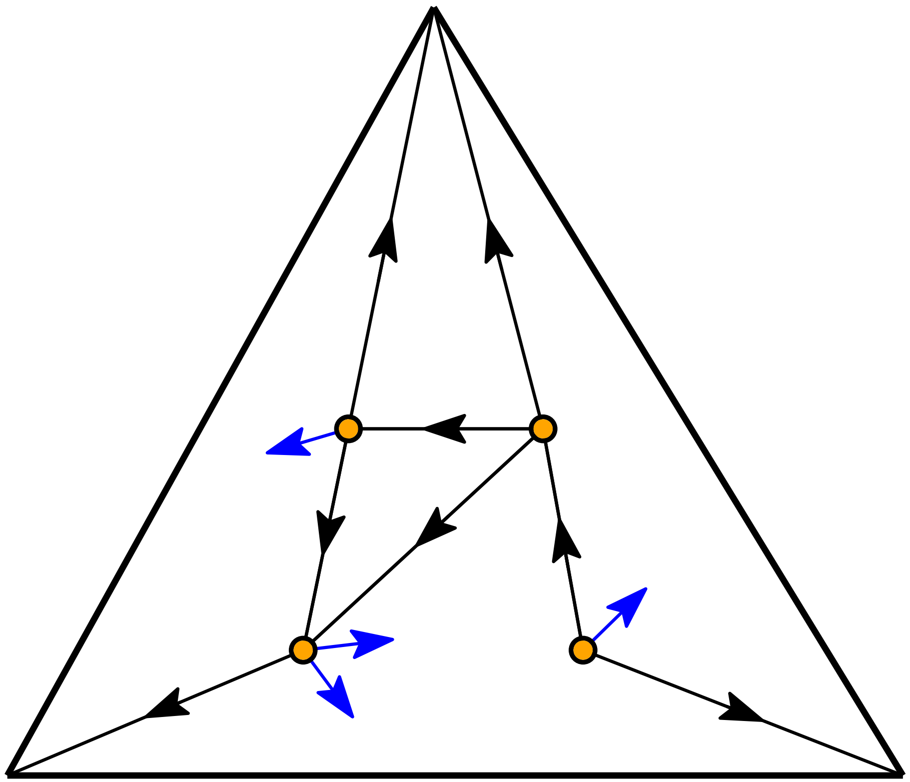

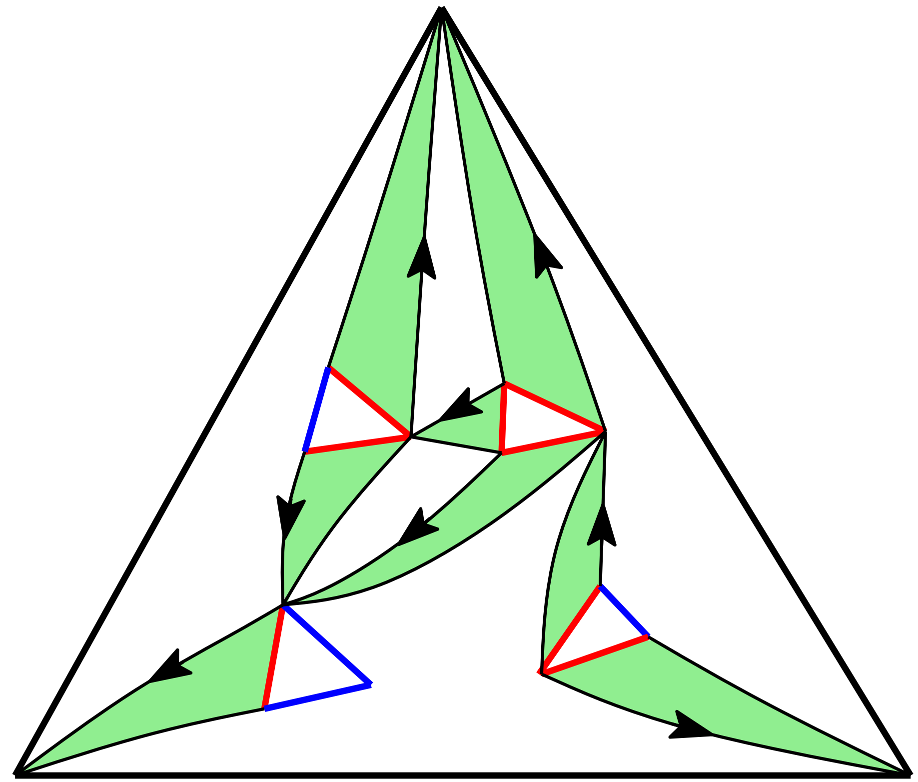

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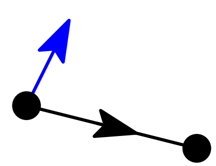

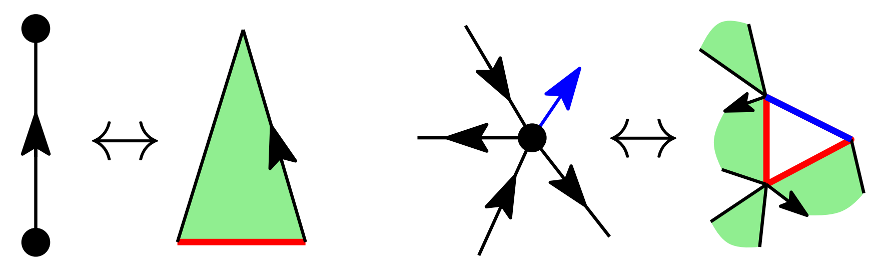

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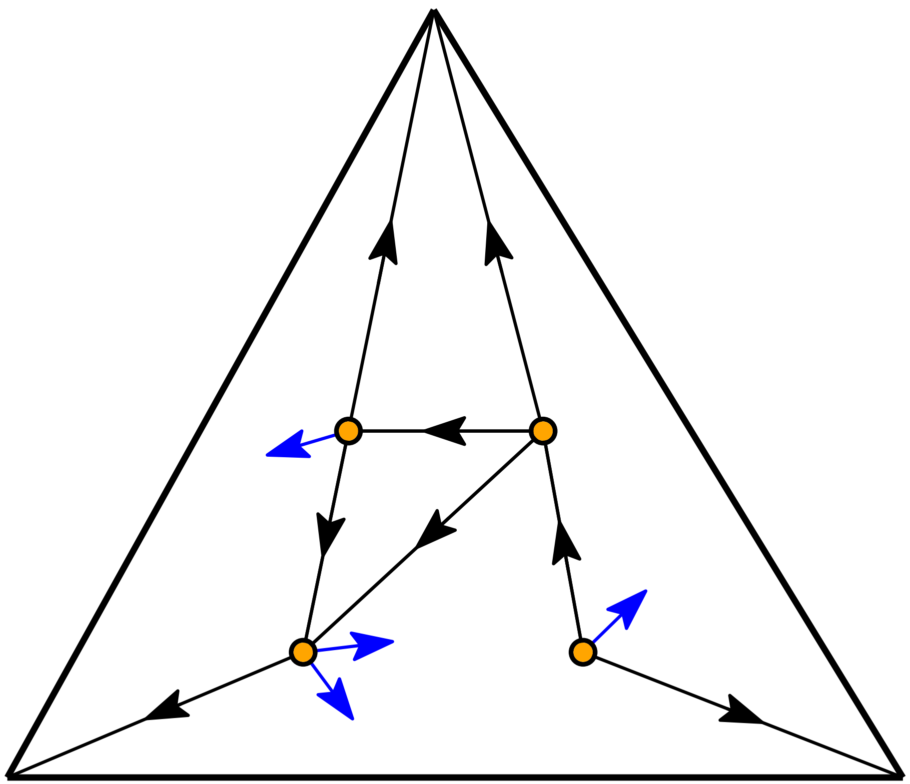

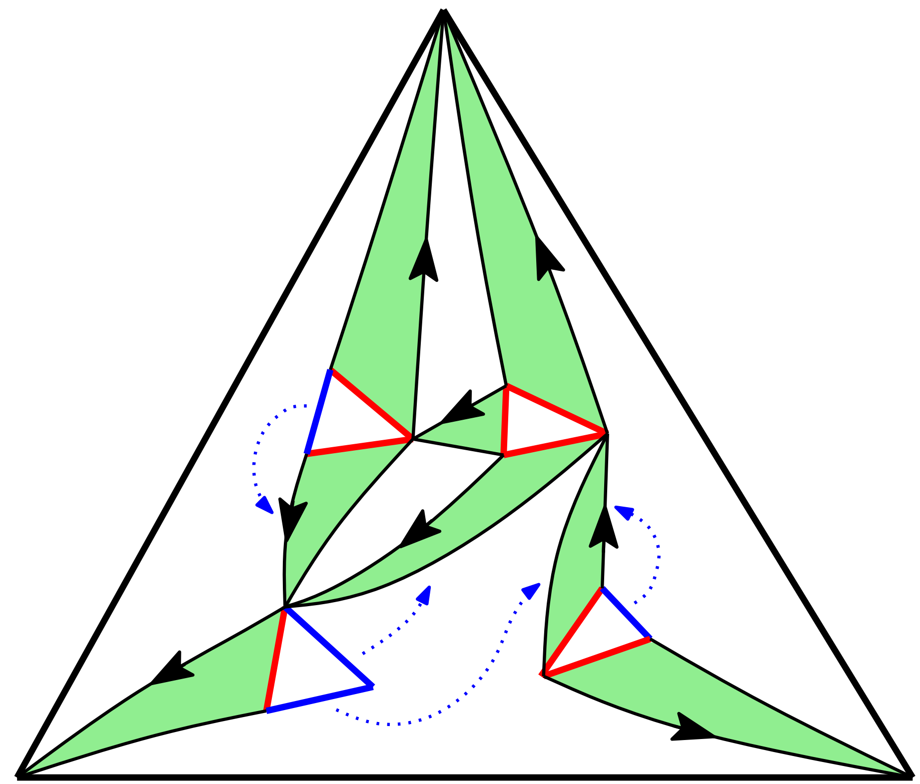

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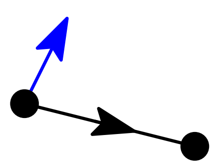
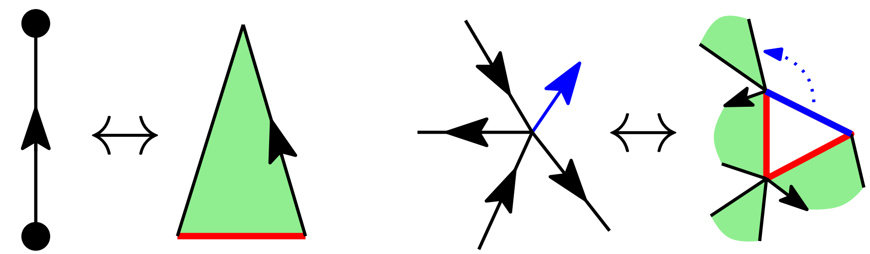
$+$ 



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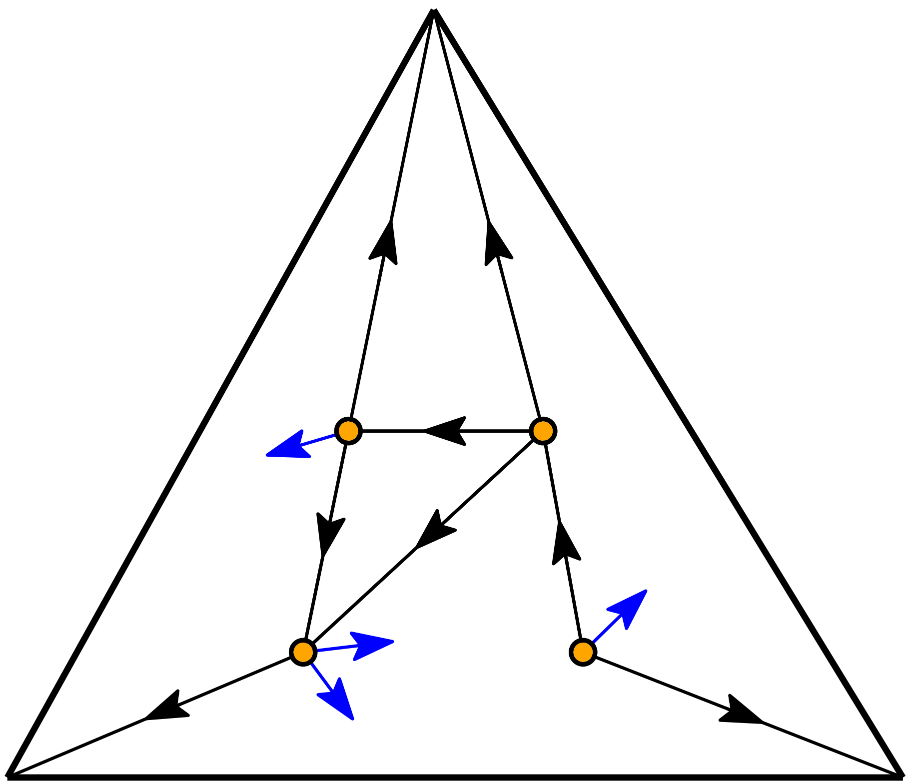

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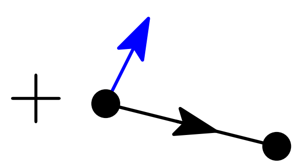

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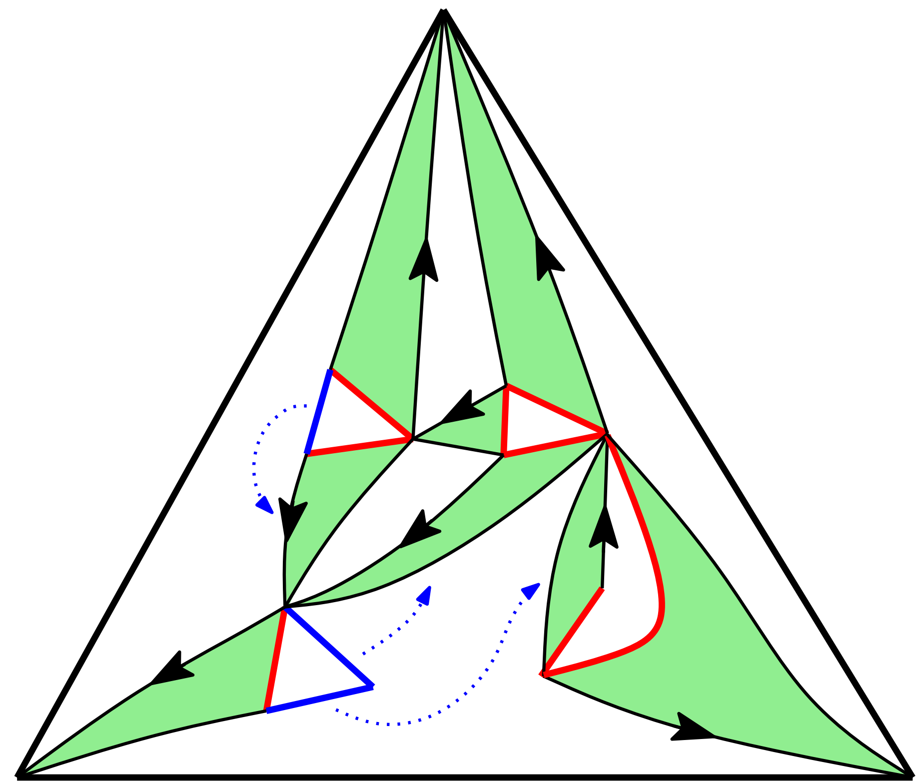

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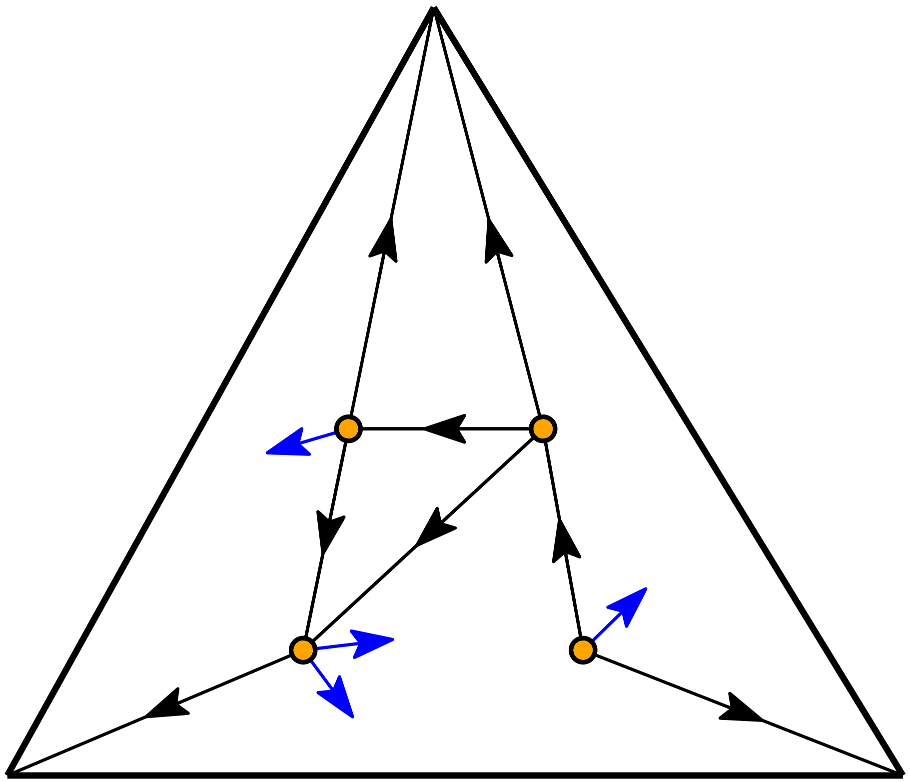

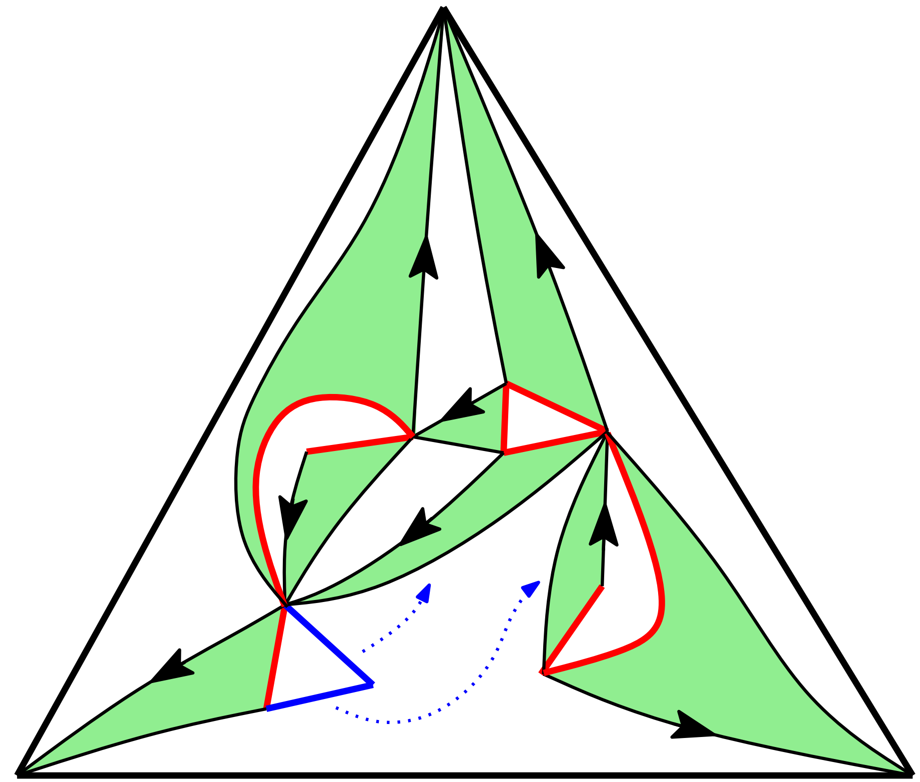

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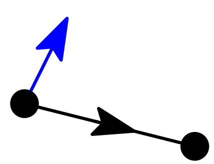
outer-accessibility + 

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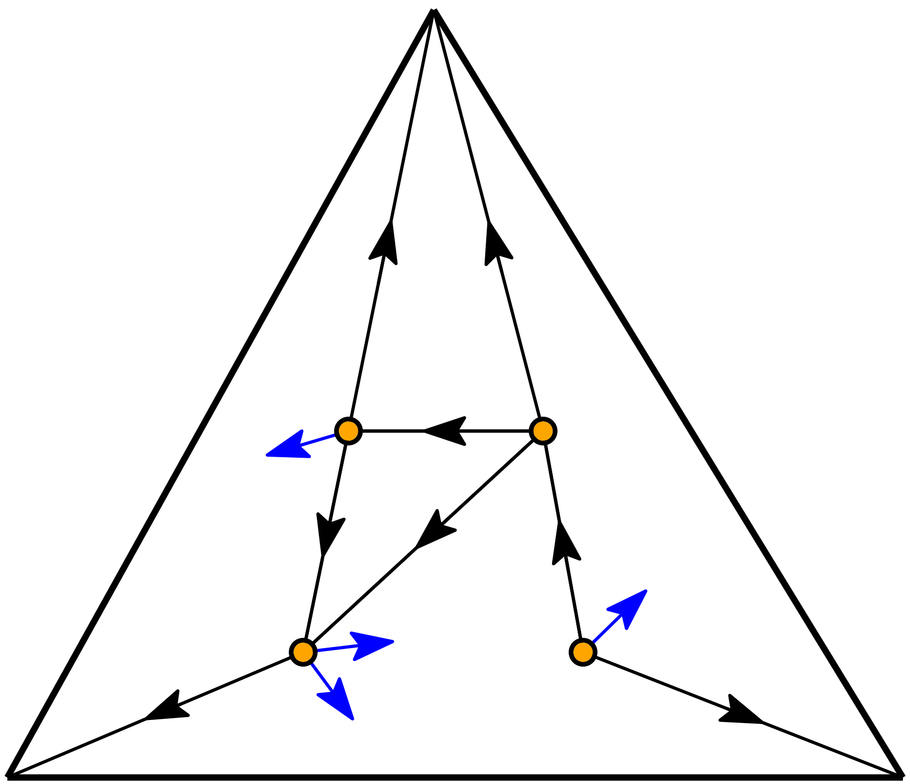

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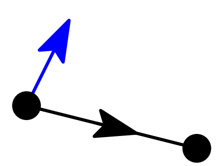
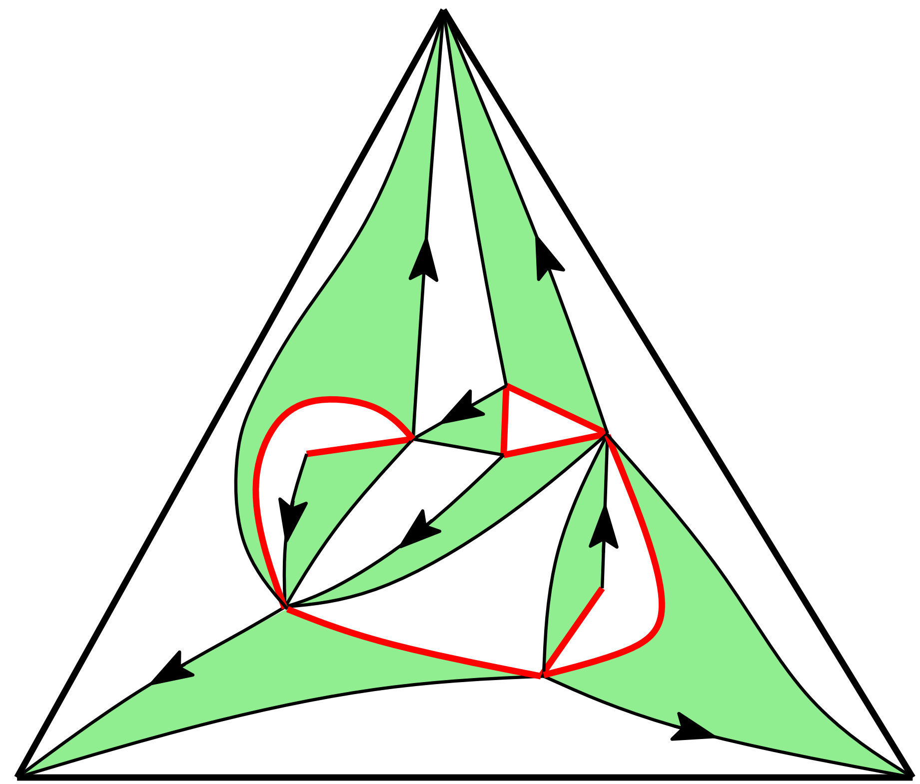
$+$ 



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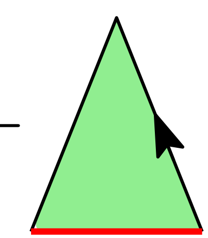
$+$ 

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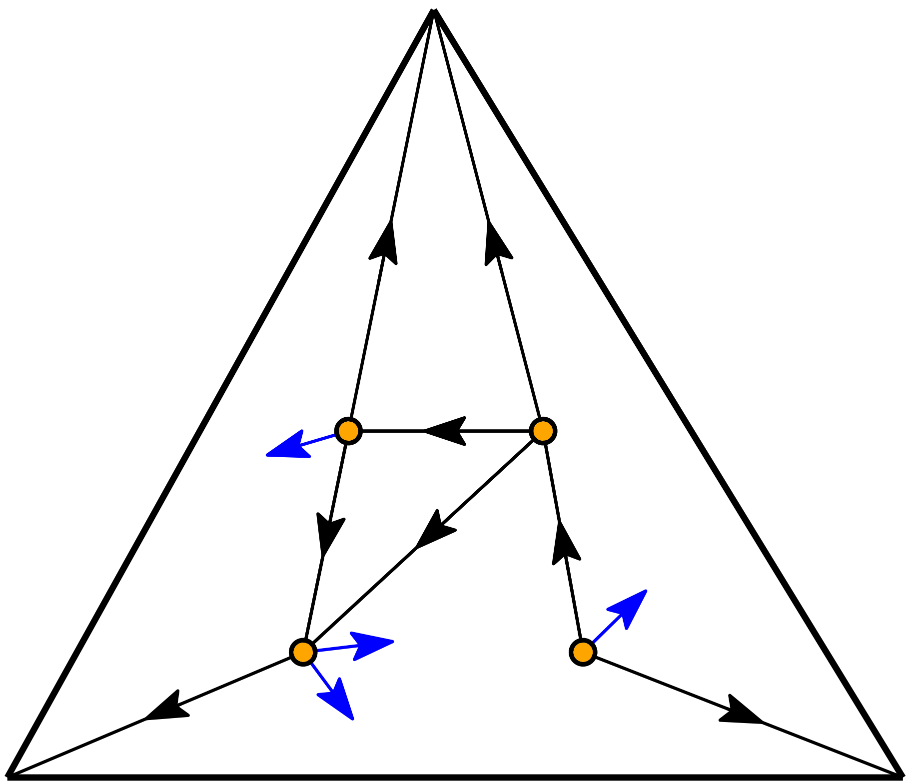

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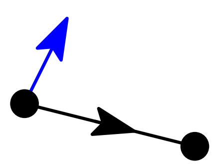
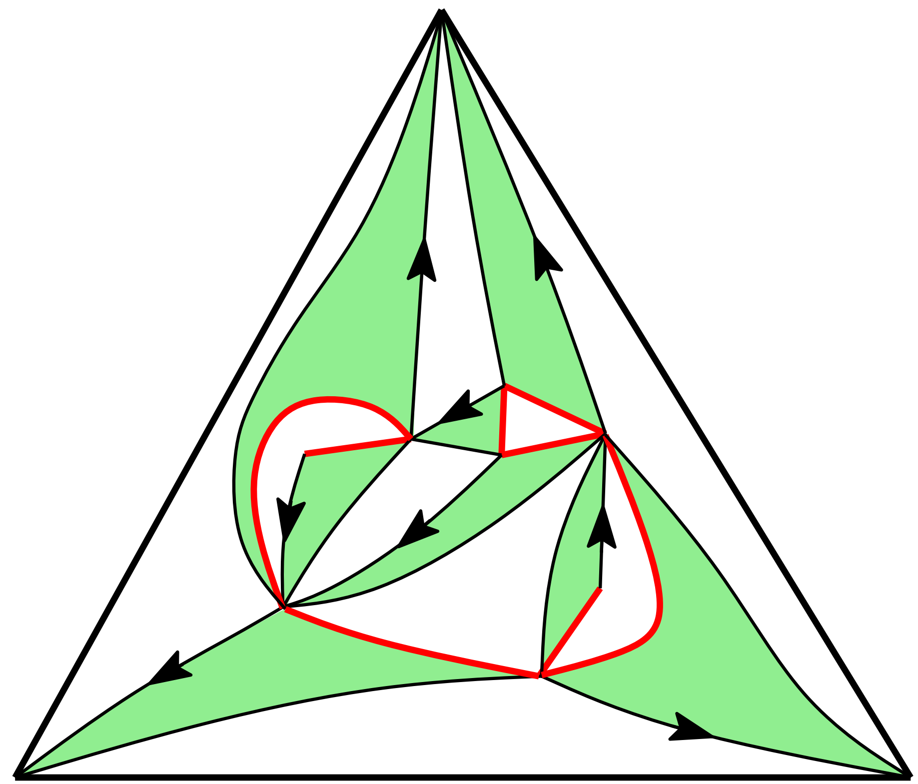
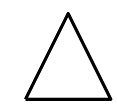
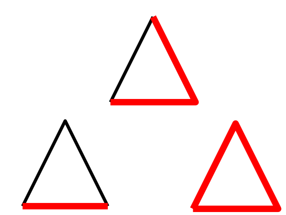
$+$ 



Face of the plane graph: 

Inner node of: the plane graph 

# Bijection for outertriangular plane graphs

**Theorem [Bernardi, C., Fusy'13] :**

This is a bijection between outertriangular plane graphs with $n+2$ edges and eulerian triangulations with $2n$ faces.

**Remark:**

$\rightarrow$ This gives a bijective proof of M. Noy's formula.

$\rightarrow$ Eulerian triangulations that arise are much easier to enumerate and generate:

- Bivariate (edges and faces) generating series
- Random sampler of plane graphs controlling the number of edges and faces.

# Distance profile of rooted plane graphs

**Distances in a random plane graph:**

Let $G$ be a plane graph rooted at $e_0$ uniform with $n$ edges

$\forall e \in E_G : d(e) =$ length of the shortest path from $e$ to $e_0$

**Profile :** $(f_k)_{k\geq 1}$, where $f_k := \frac{1}{n}\#\{e \in E_G : d(e) \leq k\}$

**Radius :** $r(G) := max(d(e), e \in E)$

**Theorem [Bernardi, C., Fusy 2014]:**

$$f_{k/(2n)^{1/4}} \xrightarrow{(d)} \text{ISE positively shifted}$$

$$r/(2n)^{1/4} \xrightarrow{(d)} \text{width of ISE (also holds for moments)}$$

**Remark:**

results similar to Chassaing–Schaeffer, with a scaling factor of $(2n)^{-1/4}$ instead of $(8n/9)^{-1/4}$.

# Proof of the convergence – 1

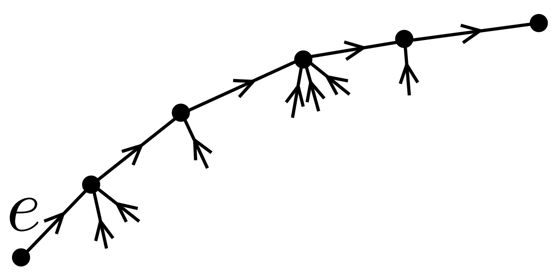
**Remark:**

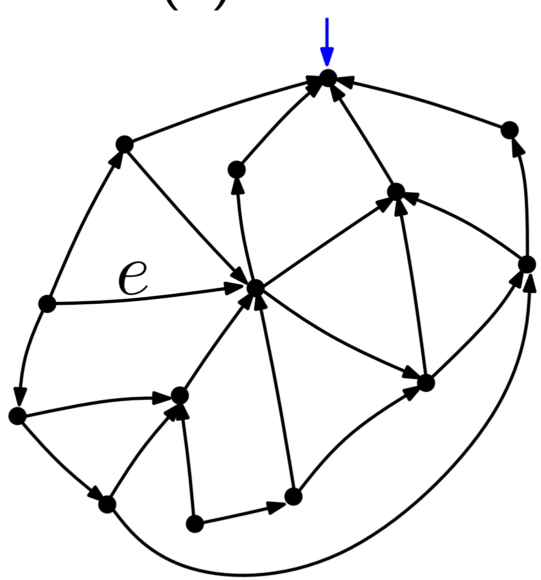
It is sufficient to prove the convergence for outertriangular plane graphs. Same results hold for both families.

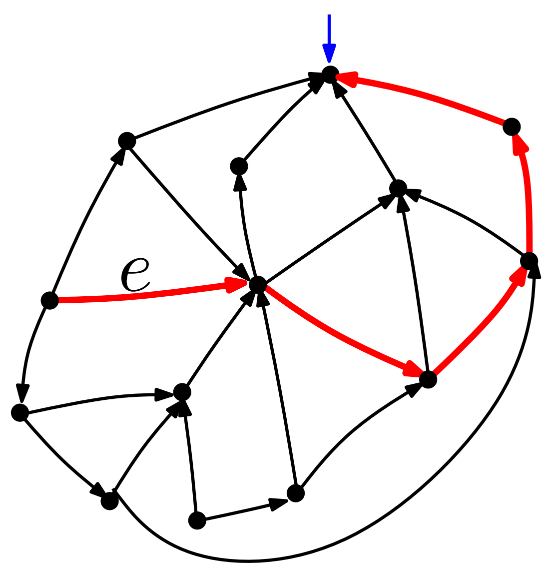
**Rightmost paths in an outertriangular plane graph:**

Let $G$ be an outertriangular plane graph with its canonical orientation For $e$ an edge of $G$, the **rightmost path** from $e$ is the unique directed path $P(e)$ starting at $e$ that turns right "as much as possible" and reaches the outer face (no loop [Bernardi '06]).

$\forall e \in E_G : \tilde{d}(e) :=$ length of P(e)



## Idea 1: Rightmost paths are quasi-geodesic in 3-orientations

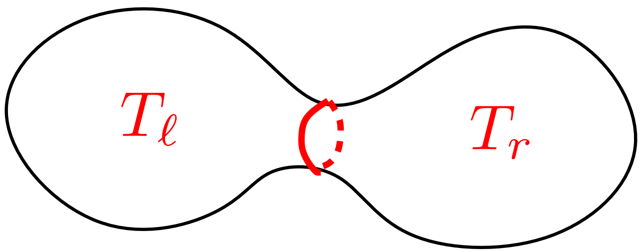
[Addario-Berry&Albenque'2013]

**Lemma:** Let $T$ a simple triangulation with $n$ vertices, $e$ an edge of $T$

If there is another path $Q$ from $e$ to the root such that

$$|Q| \leq \tilde{d}(e) - \epsilon n^{1/4}$$

then one can extract from $\tilde{d}(e) \cup Q$ a cycle $C$ of length $O(1/\epsilon)$ such that both parts $T_\ell, T_r$ after cutting along $C$ have diameter $\Omega(\epsilon n^{1/4})$



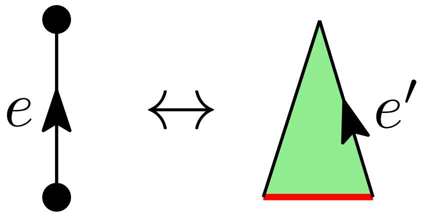

**Proposition:** Let $A_{n,\epsilon}$ the event that a random simple triangulation with $n$ vertices has an edge $e$ such that $\tilde{d}(e) - d(e) \geq \epsilon n^{1/4}$.

Then $P(A_{n,\epsilon}) \to 0$ as $n \to \infty$

From the same lemma, we can prove the analogue proposition for random simple outer-triangular maps with $n$ edges

# Proof of the convergence – 2

**Idea 2: Rightmost paths are preserved by the bijection**




- $e$ an inner edge in the plane graph maps to $e'$ an oriented edge in the eulerian triangulation
- oriented edges are left unchanged by the merging

$\Rightarrow$ $\tilde{d}(e)$ is the depth of $e'$ in the forest of the canonical orientation

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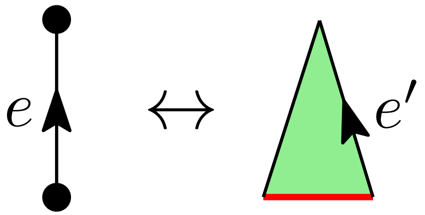


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## Idea 3: Oriented paths in the forest are "geodesic"

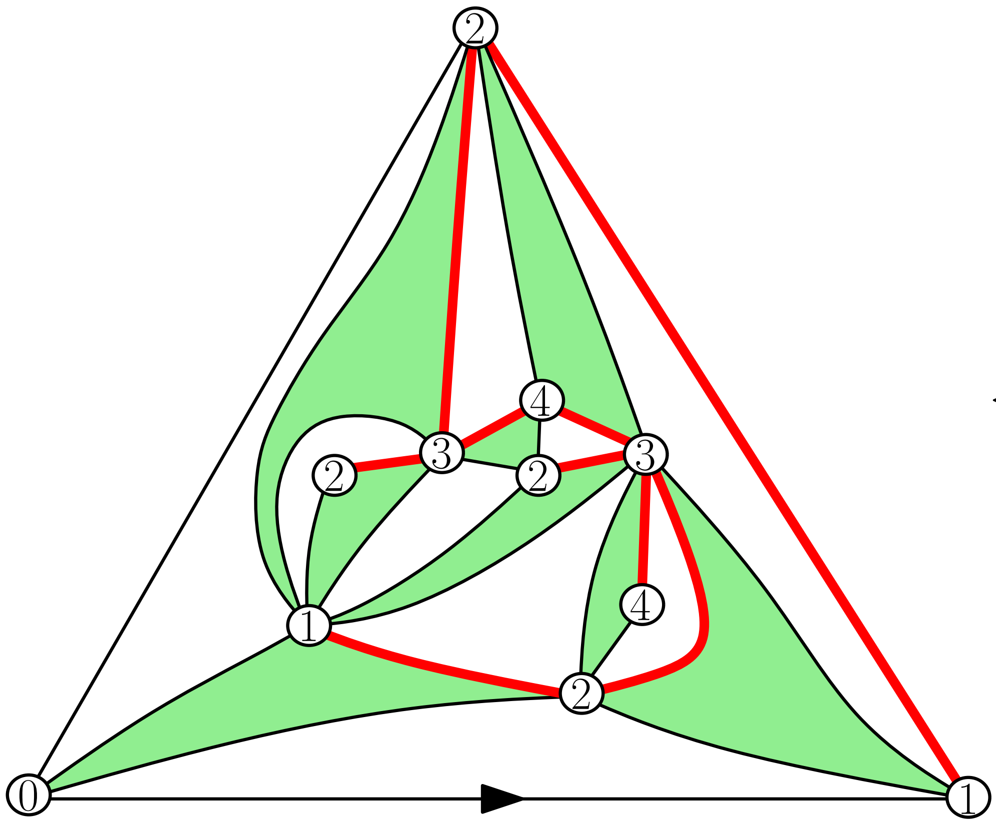
- Bousquet-Mélou–Schaeffer (2000): depth in the oriented forest of an eulerian triangulation is the length of a shortest path to the outer face having only green faces to its left.

$\Rightarrow$ $\tilde{d}(e) + \{0, 1, 2\}$ is the length of a shortest path from $e'$ to the root vertex having only green faces to its left.

# Proof of the convergence – 3

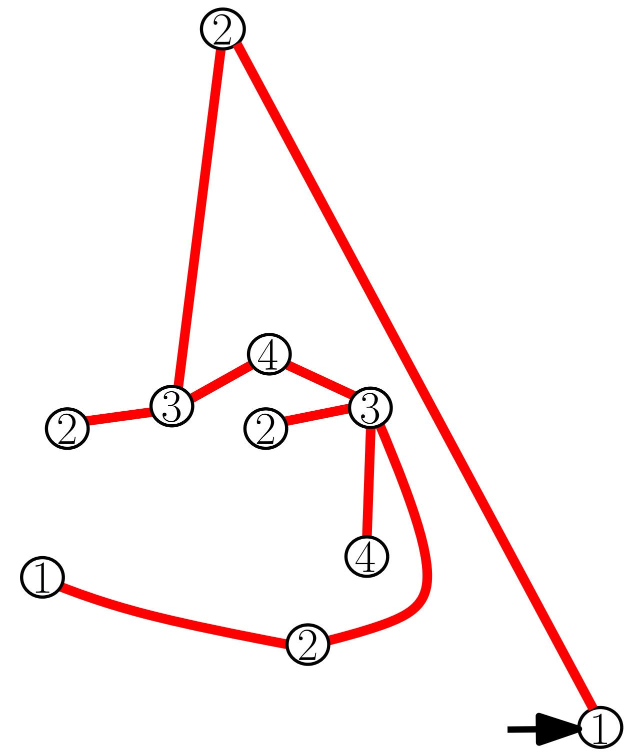
## [Chassaing–Schaeffer 2004], [Le Gall 2006]:

The distance profile (with green faces to the left) of the vertices of a rooted eulerian triangulation, rescaled by $(2n)^{-1/4}$, converges to positively shifted ISE.




Eulerian triangulation with $2n$ faces
with vertices labelled by geodesic distance

$\longleftrightarrow$




very-well-labelled tree rooted
on a vertex of minimum label

**Thank you!**