Asymptotic Analysis of the Sum of the Output of Transducers

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joint work with Clemens Heuberger and Helmut Prodinger

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Introduction

- We want to generalize
  - the sum of digits function,
  - the Hamming weight of a digit expansion, . . .
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- We want to generalize
  - the sum of digits function,
  - the Hamming weight of a digit expansion, . . .
- They can be computed by transducers.
  - They are the sum of the output.

**binary sum of digits function (Delange, 1975)**

```
0 0
1 1
```
Introduction

- We want to generalize
  - the sum of digits function,
  - the Hamming weight of a digit expansion, …
- They can be computed by transducers.
  - They are the sum of the output.
- We are interested in an asymptotic analysis:
  - central limit law
  - expected value and variance
  - probability space: \{0, 1, \ldots, N-1\} with equidistribution

**binary sum of digits function (Delange, 1975)**

\[
\begin{array}{c|c|c}
0 & 0 \\hline 1 & 1
\end{array}
\]

expected value = \( \frac{1}{2} \log_2 N + \Psi(\log_2 N) \)

with periodic, continuous \( \Psi \)
Sequences Defined by Transducers

- transducer $T$ with a finite number of states
Sequences Defined by Transducers

- transducer $\mathcal{T}$ with a finite number of states
- sequence $\mathcal{T}(n) = \text{sum of the output}$
- input: $q$-ary expansion of $n$
- read from right to left

Diagram:

```
1|0
0|1
0|0
1|1
```

Example with $n = 25$, $q = 2$:

- input: 11001
- output: 10101
- output sum: $\mathcal{T}(25) = 3$
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Example with $n = 25$, $q = 2$

| input: 11001 | output sum: $\mathcal{T}(25) =$ | output: 10101 |
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Example with $n = 25$, $q = 2$

| input: | 11001 |
| output: | 1 |

output sum: $\mathcal{T}(25) =$
Sequences Defined by Transducers

- Transducer $\mathcal{T}$ with a finite number of states
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Example with $n = 25$, $q = 2$

<table>
<thead>
<tr>
<th>Input:</th>
<th>11001</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output:</td>
<td>11</td>
</tr>
</tbody>
</table>

Output sum: $\mathcal{T}(25) =$
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Sequences Defined by Transducers

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- input: $q$-ary expansion of $n$
- read from right to left

Example with $n = 25$, $q = 2$

input: 11001
output: 1011

output sum: $\mathcal{T}(25) =$
Sequences Defined by Transducers

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- sequence $T(n) = \text{sum of the output}$
- input: $q$-ary expansion of $n$
- read from right to left

Example with $n = 25$, $q = 2$

- input: 11001
- output: 01011

output sum: $T(25) =$
Sequences Defined by Transducers

- transducer \( T \) with a finite number of states
- sequence \( T(n) = \) sum of the output
- input: \( q \)-ary expansion of \( n \)
- read from right to left

Example with \( n = 25, q = 2 \)

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<th>output</th>
<th>output sum: ( T(25) = )</th>
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<tbody>
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<td>101011</td>
<td>4</td>
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- transducer $\mathcal{T}$ with a finite number of states
- sequence $\mathcal{T}(n) = \text{sum of the output}$
- input: $q$-ary expansion of $n$
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Example with $n = 25$, $q = 2$

input: 11001
output: 101011
output sum: $\mathcal{T}(25) = 4$
Paperfolding Sequence

Definition (paperfolding sequence)

\[ n = 2^k m \text{ with } m \text{ odd}: \]

\[ f_n = \begin{cases} 
0 & \text{if } m \equiv 1 \text{ mod } 4 \\
1 & \text{if } m \equiv 3 \text{ mod } 4 
\end{cases} \]
Paperfolding Sequence

\[ f_n = \begin{cases} 0 & \text{if } m \equiv 1 \mod 4 \\ 1 & \text{if } m \equiv 3 \mod 4 \end{cases} \]
Paperfolding Sequence

\[ f = \langle 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, \ldots \rangle \]

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Paperfolding Sequence

\[ f = \langle 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, \ldots \rangle \]

Madill-Rampersad, 2013:
abelian complexity function of the paperfolding sequence:

\[ \rho(n) = \text{number of different subwords of length } n \]

different = not abelian equivalent
\[ = \text{have a different number of 1's} \]

\[ \rho(2) = 3, \text{ due to } 1, 1 \text{ and } 1, 0 \text{ and } 0, 0 \text{ (not } 0, 1) \]
Theorem (Madill-Rampersad, 2013)

\[
\begin{align*}
\rho(4n) &= \rho(2n) \\
\rho(4n + 2) &= \rho(2n + 1) + 1 \\
\rho(16n + 1) &= \rho(8n + 1) \\
\rho(16n + 5) &= \rho(4n + 1) + 2 \\
\rho(16n + 11) &= \rho(4n + 3) + 2 \\
\rho(16n + 15) &= \rho(2n + 2) + 1 \\
\rho(16n + i) &= \rho(2n + 1) + 2 \\
&\text{for } i = 3, 7, 9, 13
\end{align*}
\]
Applications

- algorithms with finite memory usage
- digit expansions
- recursions

- everything can be done by a computer
Applications

- algorithms with finite memory usage
- digit expansions
- recursions

- everything can be done by a computer

- completely $q$-additive functions
- digital sequences
- $q$-automatic sequences
Asymptotic Analysis of $\mathcal{T}(n)$

- asymptotic expected value, variance of $\mathcal{T}(n)$
- with periodic fluctuations $\Psi_1, \Psi_2$
- central limit law
- non-differentiability of $\Psi_1$
- Fourier coefficients of $\Psi_1$
Asymptotic Analysis of $\mathcal{T}(n)$

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- Fourier coefficients of $\Psi_1$

Results depend on connectivity properties of the transducer.
Connectivity Properties of the Transducer

- complete and deterministic $\leadsto$ $q$-regular

$\begin{array}{c}
\text{0|?} \\
\text{1|?}
\end{array}$

for $q = 2$
Connectivity Properties of the Transducer

- complete and deterministic $\rightsquigarrow$ $q$-regular
- final component
Connectivity Properties of the Transducer

- complete and deterministic $\iff$ $q$-regular
- final component
- finally connected
Connectivity Properties of the Transducer

- complete and deterministic $\leadsto$ \( q \)-regular
- final component
- finally connected
- period = greatest common divisor of all lengths of cycles
- final period \( p = \) least common multiple of the periods
- finally aperiodic if \( p = 1 \)

\[
\begin{align*}
\text{period: 1} \\
\text{period: 2} \\
\text{final period: } p = 2
\end{align*}
\]
Connectivity Properties of the Transducer

- complete and deterministic $\rightarrow$ $q$-regular
- final component
- finally connected
- period = greatest common divisor of all lengths of cycles
- final period $p =$ least common multiple of the periods
- finally aperiodic if $p = 1$
- reset sequence

reset sequence: 01
Central Limit Theorem

Theorem (Heuberger-K.-Prodinger, 2014)

Let \( T \) be complete, deterministic, with input alphabet \{0, 1, \ldots, q - 1\} and final period \( p \). We use the probability space \{0, 1, \ldots, N - 1\} with equidistribution for a fixed \( N \).
Central Limit Theorem

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Then \( T(n) \) has the expected value

\[
\mathbb{E}(T(n)) = e_T \log_q N + \Psi_1(\log_q N) + o(\log N)
\]

with \( e_T \) and a \( p \)-periodic, continuous function \( \Psi_1 \).
Central Limit Theorem

**Theorem (Heuberger-K.-Prodinger, 2014)**

Let $\mathcal{T}$ be complete, deterministic, with input alphabet $\{0, 1, \ldots, q - 1\}$ and final period $p$. We use the probability space $\{0, 1, \ldots, N - 1\}$ with equidistribution for a fixed $N$. Then $\mathcal{T}(n)$ has the expected value

$$E(\mathcal{T}(n)) = e^\mathcal{T} \log_q N + \Psi_1(\log_q N) + o(\log N)$$

with $\mathcal{T}$ and a $p$-periodic, continuous function $\Psi_1$.

If $\mathcal{T}$ is finally connected, then the variance is

$$\nabla(\mathcal{T}(n)) = \nu_{\mathcal{T}} \log_q N + \Psi_2(\log_q N) + o(\log N)$$

with $\nu_{\mathcal{T}} \in \mathbb{R}$ and a $p$-periodic, continuous function $\Psi_2$. 
Central Limit Theorem

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Let \( T \) be complete, deterministic, with input alphabet \( \{0, 1, \ldots, q - 1\} \) and final period \( p \). We use the probability space \( \{0, 1, \ldots, N - 1\} \) with equidistribution for a fixed \( N \). Then \( T(n) \) has the expected value

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E(T(n)) = e_T \log_q N + \Psi_1(\log_q N) + o(\log N)
\]

with \( e_T \) and a \( p \)-periodic, continuous function \( \Psi_1 \).

If \( T \) is finally connected, then the variance is

\[
\mathbb{V}(T(n)) = \nu_T \log_q N + \Psi_2(\log_q N) + o(\log N)
\]

with \( \nu_T \in \mathbb{R} \) and a \( p \)-periodic, continuous function \( \Psi_2 \).

If \( \nu_T \neq 0 \), then \( T(n) \) is asymptotically normally distributed.
Central Limit Theorem

**Theorem (Heuberger-K.-Prodinger, 2014)**

Let $\mathcal{T}$ be **complete**, **deterministic**, with input alphabet 
\{0, 1, \ldots, q − 1\} and **final period** $p$. We use the probability space 
\{0, 1, \ldots, N − 1\} with equidistribution for a fixed $N$. 
Then $\mathcal{T}(n)$ has the expected value

$$E(\mathcal{T}(n)) = e^\mathcal{T} \log_q N + \Psi_1(\log_q N) + o(\log N)$$

with $e^\mathcal{T}$ and a $p$-periodic, continuous function $\Psi_1$.

If $\mathcal{T}$ is **finally connected**, then the variance is

$$\mathbb{V}(\mathcal{T}(n)) = \nu^\mathcal{T} \log_q N + \Psi_2(\log_q N) + o(\log N)$$

with $\nu^\mathcal{T} \in \mathbb{R}$ and a $p$-periodic, continuous function $\Psi_2$.

If $\nu^\mathcal{T} \neq 0$, then $\mathcal{T}(n)$ is asymptotically normally distributed.

Also possible for higher dimensional $n$. 
Idea of the Proof

We consider the characteristic function of $\mathcal{T}(n)$

$$F(N; t) = \sum_{0 \leq n < N} e^{it\mathcal{T}(n)} = \mathbf{e}_1^\top \sum_{n=0}^{L} M^n H_{\varepsilon_n}((\varepsilon_L \ldots \varepsilon_{n+1})_q) \mathbf{u}$$

with $N = (\varepsilon_L \ldots \varepsilon_0)_q$ the $q$-ary expansion of $N$
Idea of the Proof

- We consider the characteristic function of $\mathcal{T}(n)$

\[ F(N; t) = \sum_{0 \leq n < N} e^{it\mathcal{T}(n)} = e_1^\top \sum_{n=0}^{L} M^n H_{\varepsilon_n}((\varepsilon_L \ldots \varepsilon_{n+1})_q) u \]

- with $N = (\varepsilon_L \ldots \varepsilon_0)_q$ the $q$-ary expansion of $N$
- and

\[ M = \sum_{\varepsilon=0}^{q-1} M_\varepsilon \]

with $M_\varepsilon$ a matrix with entry $e^{it\delta}$ at position $r$, $s$ if there is a transition $r \xrightarrow{\varepsilon|\delta} s$. 
Idea of the Proof

- eigenvalues of $M$ at $t = 0$
- strongly connected components
- non-final components not important
Idea of the Proof

- eigenvalues of $M$ at $t = 0$
- strongly connected components
- non-final components not important
- eigenvalues $q$ and $qe^{2\pi ik/p}$ for some $k \in \mathbb{Z}$
Idea of the Proof

\[ F(N; t) = \sum_{0 \leq n < N} e^{it\mathcal{T}(n)} = e_1^T \sum_{n=0}^L M^n H_{\varepsilon_n}((\varepsilon_L \ldots \varepsilon_{n+1})_q)u \]

- We split up \( F(N; t) \) into a part for the dominant eigenvalues and a remainder.
Idea of the Proof

\[ F(N; t) = \sum_{0 \leq n < N} e^{itT(n)} = e_1^T \sum_{n=0}^{L} M^n H_{\epsilon_n}((\epsilon_L \ldots \epsilon_{n+1})_q) u \]

- We split up \( F(N; t) \) into a part for the dominant eigenvalues and a remainder.
- Differentiation w.r.t. \( t \) \( \leadsto \) expected value and variance
- \( \Psi_1 \) and \( \Psi_2 \) are expressed by infinite sums.
Idea of the Proof

\[ F(N; t) = \sum_{0 \leq n < N} e^{itT(n)} = e_1^\top \sum_{n=0}^{L} M^n H_{\varepsilon_n}((\varepsilon_L \ldots \varepsilon_{n+1})_q)u \]

- We split up \( F(N; t) \) into a part for the dominant eigenvalues and a remainder.
- Differentiation w.r.t. \( t \) \( \leadsto \) expected value and variance
- \( \Psi_1 \) and \( \Psi_2 \) are expressed by infinite sums.
- Berry–Esseen inequality \( \leadsto \) asymptotic normality and speed of convergence \( O(\log^{-1/4} N) \)
Transducer is complete, deterministic, finally connected and the final period is 1.
Paperfolding Sequence

Transducer is complete, deterministic, finally connected and the final period is 1.

\[
\mathbb{E}(\rho(n)) = \frac{8}{13} \log_2 N + \Psi_1(\log_2 N) + o(\log N)
\]

\[
\mathbb{V}(\rho(n)) = \frac{432}{2197} \log_2 N + \Psi_2(\log_2 N) + o(\log N)
\]

\(\Psi_1\) and \(\Psi_2\) are 1-periodic and continuous.
Paperfolding Sequence

Transducer is complete, deterministic, finally connected and the final period is 1.

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E(\rho(n)) = \frac{8}{13} \log_2 N + \Psi_1(\log_2 N) + o(\log N)
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\]

\(\Psi_1\) and \(\Psi_2\) are 1-periodic and continuous.

\(\sim\) asymptotically normally distributed
Fourier Coefficients

**Theorem (Heuberger-K.-Prodinger, 2014)**

Assume that the transducer is *finally connected, finally aperiodic*. Then the Fourier coefficients of $\Psi_1$ are

$$c_0 = -\frac{e^T}{2} + b_0 + \frac{1}{\log q} b_1 + \text{Res}_{z=1} H(z),$$

$$c_\ell = \frac{e^T}{\chi_\ell \log q} + \frac{1}{(1 + \chi_\ell) \log q} b_1 + \frac{1}{1 + \chi_\ell} \text{Res}_{z=1+\chi_\ell} H(z)$$

for $\ell \neq 0$, $\chi_\ell = \frac{2\pi i \ell}{\log q}$ and $b_0, b_1 \in \mathbb{R}$.

The Dirichlet function $H(z)$ satisfies an infinite recursion.
Paperfolding Sequence: $\Psi_1$

partial Fourier series with 24 Fourier coefficients
empirical values
Non-Differentiability

**Theorem (Heuberger-K.-Prodinger, 2014)**

Assume that the transducer has a reset sequence, the output alphabet is $\mathbb{Z}$ and that $e_T \notin \mathbb{Z}$.

Then $\Psi_1$ is non-differentiable for every $x \in \mathbb{R}$.

- often used reset sequence: $0 \cdots 0$
- paperfolding sequence: reset sequence 00001
  $\leadsto$ non-differentiable $\Psi_1$
Theorem (Heuberger-K.-Prodinger, 2014)

Assume that the transducer has a reset sequence, the output alphabet is $\mathbb{Z}$ and that $e_T \notin \mathbb{Z}$.
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- often used reset sequence: $0 \cdots 0$
- paperfolding sequence: reset sequence 00001
  $\leadsto$ non-differentiable $\Psi_1$
- The existence of a reset sequence is not guaranteed.
Recursions and Transducers

**Paperfolding sequence**

\[
\begin{align*}
\rho(4n) &= \rho(2n) & \rho(16n + 11) &= \rho(4n + 3) + 2 \\
\rho(4n + 2) &= \rho(2n + 1) + 1 & \rho(16n + 15) &= \rho(2n + 2) + 1 \\
\rho(16n + 1) &= \rho(8n + 1) & \rho(16n + i) &= \rho(2n + 1) + 2 \\
\rho(16n + 5) &= \rho(4n + 1) + 2 & \text{for } i = 3, 7, 9, 13
\end{align*}
\]

Recursions similar to the paperfolding sequence:

\[
\rho(q^k n + \lambda) = \rho(q^{k_\lambda} n + r_\lambda) + t_\lambda \quad \text{for } 0 \leq \lambda < q^k
\]

with \(0 \leq k_\lambda < k\)
Recursions similar to the paperfolding sequence:

\[ \rho(q^n + \lambda) = \rho(q^{\lambda} n + r_\lambda) + t_\lambda \quad \text{for} \quad 0 \leq \lambda < q^k \]

Construction:

- assume here \( 0 \leq r_\lambda < q^{\lambda} \)
- tree of depth \( k - 1 \)

for \( q = 2, k = 3 \)
Recursions and Transducers

Recursions similar to the paperfolding sequence:

$$\rho(q^n + \lambda) = \rho(q^{\lambda n} + r) + t\lambda \quad \text{for} \quad 0 \leq \lambda < q^k$$

Construction:

- assume here $0 \leq r_\lambda < q^{k\lambda}$
- tree of depth $k - 1$
- transitions

$$(\ell \mod q^{k-1}) \xrightarrow{c|t\lambda} (r_\lambda \mod q^{k\lambda})$$

with $\lambda = q^{k-1}c + \ell$

for $q = 2, k = 3$
Conclusion

- We can asymptotically analyze sequences.
- The sequence is the output sum of a transducer.
- Joint generalization of many results.
- Everything is implemented (e.g. in Sage).
- We obtain different results, depending on the properties of the transducer.
Conclusion

- We can asymptotically analyze sequences.
- The sequence is the output sum of a transducer.
- Joint generalization of many results.
- Everything is implemented (e.g., in Sage).
- We obtain different results, depending on the properties of the transducer.

- What happens if assumptions are not satisfied?
  - no reset sequence: differentiability of $\Psi_1$?
  - not finally connected: central limit law?