On Symmetry of Uniform and Preferential Attachment Graphs

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A graph has a **symmetry** if there are two nodes that have *the same global view of the (unlabeled) graph*.

![Graphs with symmetry](image)

**Definition (Automorphism, automorphism group)**

An **automorphism** of a graph $G$ on $n$ vertices is a permutation $\pi : [n] \to [n]$ such that $\{u, v\} \in E(G) \iff \{\pi(u), \pi(v)\} \in E(G)$. The set of automorphisms of $G$ with the operation of function composition is called the **automorphism group** $\text{Aut}(G)$ of $G$.

An automorphism is an isomorphism that results in **the same labeled graph**.
Application: Counting structures

Motivating question: How many unlabeled graphs of size $n$ are there? Unlabeled $d$-regular graphs?

Size of the isomorphism class of $G$: $|S(G)| = \frac{n!}{|\text{Aut}(G)|}$.

Figure:

$\text{Aut}(G_1) = \{\text{ID}, (1432), (13), (24), (12)(34), (13)(24), (14)(23), (1234)\}$,
$\text{Aut}(G_2) = \{\text{ID}, (13), (24), (13)(24)\}$.

Erdős and Rényi: Almost all graphs are asymmetric (so sizes of isomorphism classes are almost all $n$!).
Application: Structure compression

Motivating question: How much more can we compress a graph if we can throw away the labels?

Choi and Szpankowski (2012):

**Theorem**

Provided all isomorphic graphs are equiprobable,

\[ H_G = H_S + \log n! - \sum_{s \in S} P(s) \log |\text{Aut}(s)|. \]

Here, \( H_G \) is the entropy of the distribution on labeled graphs \( \mathcal{G} \), and \( H_S \) is the entropy of the distribution on structures \( \mathcal{S} \) induced by \( \mathcal{G} \). So more symmetry \( \implies \) less of a difference in compressibility between labeled and unlabeled graphs.
Defect: A tool for proving asymmetry

Let $N(u)$ denote the neighbors of $u$.

- Defect of a vertex: $D_\pi(u) := |N(\pi(u)) \Delta \pi(N(u))|$.  
- Defect of a permutation: $D_\pi(G) := \max_{u \in V(G)} D_\pi(u)$.  
- Defect of a graph: $D(G) := \min_{\pi \neq \text{ID}} D_\pi(G)$.

Defect-based criterion for asymmetry:

$G$ is asymmetric $\iff D(G) > 0 \iff \forall \pi \neq \text{ID}, \exists u \text{ s.t. } D_\pi(u) > 0.$

Figure: $D_\pi(2) = |\{2, 4\} \Delta \{1, 2, 4, 5\}| = 2$, $D_\pi(G) \geq 2$, $D(G) = 0$. 
Prior work

- **Asymmetric Graphs** (Erdős and Rényi, 1963): For fixed $p$, $G(n, p)$ is asymmetric with high probability.

- **The Asymptotic Number of Unlabelled Regular Graphs**, (Bollobás, 1982): Random $d$-regular graphs (for $d \geq 3$ fixed) are asymptotically asymmetric.

- **On the Asymmetry of Random Regular Graphs and Random Graphs** (Kim, Sudakov, Vu, 2002): If $p \gg \frac{\log n}{n}$ and $1 - p \gg \frac{\log n}{n}$, we have, almost surely,
  
  
  $D(G(n, p)) = (2 - o(1)) np(1 - p)$.

  If $\log n \ll d$ and $n - d \gg \log n$, then, almost surely,
  
  $D(G(n, d)) = (2 + o(1))d \left(1 - \frac{d}{n}\right)$.
Networks in the real world exhibit a power law degree distribution.

Barabási & Albert: This could arise from a rich get richer mechanism.

Bollobás & Riordan: BA’s description is mathematically imprecise. The preferential attachment model is born!
A preferential attachment graph $\mathcal{PA}(n, m)$ on $n$ vertices, with $m$ choices per vertex:
Main results: symmetry

Theorem (Symmetry results for \( m = 1, 2 \))

Fix \( m = 1, 2 \), and let \( G_n \sim UA(n, m) \) or \( G_n \sim PA(n, m) \). Then there exists a constant \( C > 0 \) such that, for \( n \) sufficiently large,

\[
\Pr[|\text{Aut}(G_n)| > 1] > C.
\]

For the uniform attachment model, the result for \( m = 1 \) can be strengthened to symmetry with high probability.
Asymmetry conjecture

**Conjecture**

For fixed $m \geq 3$, and either $G_n \sim \mathcal{P}A(n, m)$ or $G_n \sim \mathcal{U}A(n, m)$,

$$\lim_{n \to \infty} \Pr[|\text{Aut}(G_n)| = 1] = 1.$$ 

- Proving asymmetry is challenging (it is a **global** property of a graph, while symmetry is **local**).
- There is ample empirical evidence for the conjecture, and we add some more via **defect**.
Empirical evidence for asymmetry

Numerical defect vs graph size for the Uniform Attachment model

Figure: Empirical graph defect for graphs up to 1000 nodes.
Let $\pi$ be a nontrivial permutation in $S_n$, and let $u \in [n]$. Then we define

$$\omega(\pi, u) = \min\{u, \pi(u)\},$$

and

$$\omega(\pi) = \min\{v \in [n] | \pi(v) \neq v\}.$$
Main results: expected defect, weak asymmetry

**Theorem (Expected vertex defect)**

Fix $m \in \mathbb{N}$ in the uniform attachment model. For $n$ sufficiently large, $\pi \neq \text{ID}$, $\pi \in S_n$, and $u \in [n]$ not fixed by $\pi$,

$$
\log \left( \frac{n}{\max\{\omega(\pi, u) + 2, (2m + 2)\}} \right) \leq \mathbb{E}[D_{\pi}(u)] \leq 1 + 4m \left( 2 + \log \left( \frac{n}{\omega(\pi, u)} \right) \right).
$$

**Theorem (Weak asymmetry)**

Fix $m \geq 1$ and consider a sequence of graphs in the uniform attachment model $G_n \sim \mathcal{U}A(n, m)$. Let $\{\pi_n\}_{n=1}^{\infty}$, $\pi_n \in S_n - \{\text{ID}\}$, and, for each $n$, let $u_n = \omega(\pi_n)$. Then

$$
\Pr[D_{\pi_n}(u_n) = 0] \xrightarrow{n \to \infty} 0.
$$
Main idea: Lower bound the probability of a small subgraph that implies symmetry.
Proof sketch: more details on $m = 2$

Goal: Consider two nodes that *make the same choices* and are *unchosen*.

- Probability that there are $u, v > \frac{n}{2}$ such that
  $\{c_{u,1}, c_{u,2}\} = \{c_{v,1}, c_{v,2}\} \subset [n/2]$: bounded below by $C > 0$, by a birthday paradox argument: $\Theta(n^2)$ birthdays, $\Theta(n^2)$ people.

- Condition on lexicographic ordering of pairs of vertices $> \frac{n}{2}$ and on the event that two such vertices pick the same pair. Conclude that such a pair is unchosen with positive probability.
Future work

1. Asymmetry for growing $m_t$ as a function of time: defect-based proof works in the uniform attachment case, and possibly with preferential attachment.

2. Asymmetry for constant $m \geq 3$?

3. Study the structure of the automorphism group for cases where symmetry has positive probability.