Random graphs from a minor-closed class

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Question

Let $\mathcal{A}$ be a class of (simple) graphs closed under isomorphism, eg the class $\mathcal{P}$ of planar graphs.

$\mathcal{A}_n$ is the set of graphs in $\mathcal{A}$ on vertices 1, \ldots, $n$.

$R_n \in_u \mathcal{A}$ means that $R_n$ is picked uniformly at random from $\mathcal{A}_n$.

What are typical properties of $R_n$?

usually a giant component? probability of being connected? many vertices of degree 1? size of the 2-core?

Can we learn anything useful for the design or analysis of algorithms?
Generating functions

Given a class $\mathcal{A}$ of graphs, $A(x)$ denotes the exponential generating function (egf) $\sum_n |\mathcal{A}_n| x^n / n!$. Also $\rho = \rho(A) = \rho(\mathcal{A})$ is the radius of convergence.

For suitable classes of graphs, we can relate the egfs (or two variable versions) of all graphs, connected graphs, 2-connected graphs and 3-connected graphs.

If we know enough about the 3-connected graphs (as we do for planar graphs, thanks to Tutte and others) then we may be able to extend to all graphs.

Let us leave that for now and proceed in greater generality.
Minors

$H$ is a **minor** of $G$ if $H$ can be obtained from a subgraph of $G$ by edge-contractions.

$\mathcal{A}$ is **minor-closed** if

$$G \in \mathcal{A}, \ H \text{ a minor of } G \quad \Rightarrow \quad H \in \mathcal{A}$$

Examples:
- forests, series-parallel graphs, and more generally graphs of treewidth $\leq k$;
- outerplanar graphs, planar graphs, and more generally graphs embeddable on a given surface;
- graphs with at most $k$ (vertex) disjoint cycles.
Minors

\( \text{Ex}(\mathcal{H}) \) is the class of graphs with no minor a graph in \( \mathcal{H} \). For example: series-parallel graphs = \( \text{Ex}(K_4) \), planar graphs = \( \text{Ex}(\{K_5, K_{3,3}\}) \), graphs with no two disjoint cycles = \( \text{Ex}(2C_3) \).

Easy to see that: \( \mathcal{A} \) is minor-closed iff \( \mathcal{A} = \text{Ex}(\mathcal{H}) \) for some class \( \mathcal{H} \).

Robertson and Seymour’s graph minors theorem (once Wagner’s conjecture) is that if \( \mathcal{A} \) is minor-closed then \( \mathcal{A} = \text{Ex}(\mathcal{H}) \) for some finite class \( \mathcal{H} \).

The unique minimal such \( \mathcal{H} \) consists of the excluded minors for \( \mathcal{A} \).
Minors

Mostly we shall assume that $\mathcal{A}$ is minor-closed and proper (that is, not empty and not all graphs). For such $\mathcal{A}$, a result of Mader says:
there is a $c = c(\mathcal{A})$ such that the average degree of each graph in $\mathcal{A}$ is at most $c$.

Thus our graphs are sparse. For $\text{Ex}(K_t)$ the maximum average degree is of order $t\sqrt{\log t}$ (Kostochka, Thomason).

Call $\mathcal{A}$ small if $\rho(\mathcal{A}) > 0$, that is $\exists c$ such that $|\mathcal{A}_n| \leq c^n n!$.

Norine, Seymour, Thomas and Wollan (2006); and Dvorák and Norine (2010) showed that:
Each (proper) minor-closed graph class $\mathcal{A}$ is small.
Decomposable

If a graph is in $\mathcal{A}$ if and only if each component is, then we call $\mathcal{A}$ decomposable.

For example the class of planar graph is decomposable but the class of graphs embeddable on the torus is not.

A minor-closed class is decomposable iff each excluded minor is connected.

Let $\mathcal{A}$ be a decomposable class of graphs; and let $\mathcal{C}$ consist of the connected graphs in $\mathcal{A}$, with egf $C(x)$. The exponential formula says that $A(x) = e^{C(x)}$. 
Bridge-addable and addable

A is **bridge-addable** if whenever $G \in \mathcal{A}$ and $u$ and $v$ are in different components of $G$ then $G + uv \in \mathcal{A}$.

A is **addable** if it is decomposable and bridge-addable.

A minor-closed class $\mathcal{A}$ is addable iff each excluded minor is 2-connected.

$\mathcal{G}^S$ is bridge-addable but **not** decomposable (and so not addable) except in the planar case.
Bridge-addability and being connected

From McD, Steger and Welsh (2005):

**Lemma**

*If \( \mathcal{A} \) is bridge-addable and \( R_n \in u \mathcal{A} \) then*

\[
\mathbb{P}(R_n \text{ is connected}) \geq \frac{1}{e}.
\]

For trees \( \mathcal{T} \) and forests \( \mathcal{F} \), \( |\mathcal{T}_n| = n^{n-2} \) and \( |\mathcal{F}_n| \sim e^{\frac{1}{2}} n^{n-2} \). Thus for \( F_n \in u \mathcal{F} \),

\[
\mathbb{P}(F_n \text{ is connected}) \sim e^{-\frac{1}{2}}.
\]
Bridge-addability and being connected

McD, Steger and Welsh (2006) conjectured:

Conjecture

If $\mathcal{A}$ is bridge-addable then $\mathbb{P}(R_n \text{ is connected}) \geq e^{-\frac{1}{2} + o(1)}$.

Balister, Bollobás and Gerke (2008, 2010) give an asymptotic lower bound of $e^{-0.7983}$. Norine (2013) improves this to $e^{-2/3}$, but the full conjecture is still open.

Addario-Berry, McD and Reed (2012), and Kang and Panagiotou (2013) prove the conjecture if $\mathcal{A}$ is also closed under deleting bridges, that is if $\mathcal{A}$ is bridge-alterable.
Introduction
Properties of graph classes
Counting
More generally

Bridge-alterability and connectivity

Here is a natural strengthening of the last conjecture, see eg Balister, Bollobás and Gerke (2010).

**Conjecture**

Let $\mathcal{A}$ be bridge-addable, $R_n \in_u \mathcal{A}$ and $F_n \in_u \mathcal{F}$. Then

$$\mathbb{P}(R_n \text{ is connected}) \geq \mathbb{P}(F_n \text{ is connected}).$$

(Recall that $\mathbb{P}(F_n \text{ is connected}) \sim e^{-\frac{1}{2}}$.)
Bridge-alterability and connectivity

The result below (2013) gives a weakened form of the conjecture.

**Theorem**

Let \( A \) be bridge-alterable, \( R_n \in_u A \), and \( F_t \in_u F \) for \( t = 1, 2, \ldots \). Then

\[
P(R_n \text{ is connected}) \geq \min_{n/3 < t \leq n} P(F_t \text{ is connected}).
\]

The value \( n/3 \) can be increased towards \( n/2 \).
Big component

The **big component** $\text{Big}(G)$ of a graph $G$ is the (lex first) component with most vertices.

The **fragment** ‘left over’, $\text{Frag}(G)$, is the subgraph induced on the vertices not in the big component.

Write $\text{frag}(G)$ for $\nu(\text{Frag}(G))$.

**Theorem**

*If $\mathcal{A}$ is bridge-addable then $\mathbb{E}[\text{frag}(R_n)] < 2$.*

Thus $\text{Big}(R_n)$ is giant!
Growth constant

\(A\) has a growth constant \(\gamma\) if

\[
\left(\frac{|A_n|}{n!}\right)^{1/n} \to \gamma \quad \text{as} \quad n \to \infty,
\]

that is, if

\[|A_n| = (\gamma + o(1))^n \ n!\].

\(A\) has growth constant \(\gamma \implies A(\lambda)\) has radius of convergence \(\rho = 1/\gamma\).

If \(A\) is decomposable, then the exponential formula shows that \(A\) and \(C\) have the same radius of convergence.

Observe that: \(A\) contains all paths \(\implies \rho \leq 1\).

Bernardi, Noy and Welsh 2010: if \(A\) does not contain all paths then \(\rho = \infty\) (assuming \(A\) is monotonic).
When is there a growth constant?

small and addable

McD, Steger and Welsh (2005):

Lemma

\[ \mathcal{A} \text{ small and addable } \Rightarrow \exists \text{ growth constant } \gamma(\mathcal{A}) \]

Proof. Since \( \mathcal{A} \) is bridge-addable, \( \mathbb{P}(R_n \text{ is connected}) \geq 1/e \).

Since also \( \mathcal{A} \) is decomposable

\[
|\mathcal{A}_{a+b}| \geq \binom{a+b}{a} \frac{|\mathcal{A}_a|}{e} \frac{|\mathcal{A}_b|}{e} \frac{1}{2}
\]

and so \( f(n) = \frac{|\mathcal{A}_n|}{2e^2 n!} \) satisfies \( f(a + b) \geq f(a) \cdot f(b) \); that is, \( f \) is supermultiplicative. Now use ‘Fekete’s lemma’ to show that

\[
f(n)^{1/n} \to \sup_k f(k)^{1/k} < \infty.
\]
When is there a growth constant?

minor-closed and addable – and $G^S$

**Theorem**

*Each addable proper minor-closed class $A$ has a growth constant $\gamma(A)$.*

In particular the class $\mathcal{P}$ of planar graphs has a growth constant. For any surface $S$ other than the plane, the class $G^S$ of graphs embeddable on $S$ is bridge-addable but not addable. However, we could show (2008) that $G^S$ has the same growth constant as $\mathcal{P}$. (We now know much more, indeed asymptotic formulae.)

Bernardi, Noy and Welsh (2010) asked:

*does every proper minor-closed class of graphs have a growth constant?*
Having a growth constant yields ..

Pendant copies theorem - introduction

Let $H$ be a connected graph with a root vertex. $G$ has a pendant copy of $H$ if $G$ has a bridge $e$ with $H$ at one end, where $e$ is incident with the root of $H$.

$H$ is freely attachable to $\mathcal{A}$ if whenever we have a graph $G$ in $\mathcal{A}$ and a disjoint copy of $H$, and we add an edge between a vertex in $G$ and the root of $H$, then the resulting graph must be in $\mathcal{A}$.

For an addable minor-closed class $\mathcal{A}$, the class of freely attachable graphs is the class of connected graphs in $\mathcal{A}$.

For $\mathcal{G}^S$, the class of freely attachable graphs is the class of connected planar graphs.
Pendant copies theorem

**Theorem**

Let $\mathcal{A}$ have a finite positive growth constant, and let $H$ be freely attachable to $\mathcal{A}$. Let $R_n \in_u \mathcal{A}$. Then there exists $\alpha > 0$ such that

$$\Pr(R_n \text{ has } < \alpha n \text{ pendant copies of } H) = e^{-\Omega(n)}.$$ 

Often this shows that there are linear numbers of vertices of each degree, and exponentially many automorphisms.

For $R_n \in_u \mathcal{P}$, whp $\omega(R_n) = 4$ and so $\chi(R_n) = 4$.

Hadwiger’s Conjecture being false says that for some $k$, there is a graph $G \in \text{Ex}(K_k)$ with $\chi(G) \geq k$.

But then for $R_n \in_u \text{Ex}(K_k)$, wvhp $\chi(R_n) \geq k$. All but an exponentially small proportion of $(K_k)$ are counterexamples!
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Pendant copies theorem
searching for a subgraph

Let $\mathcal{A}$ have a growth constant, and let the connected graph $H$ be freely attachable. For example, let $\mathcal{A}$ be addable and minor-closed and let $H \in \mathcal{A}$; or let $\mathcal{A}$ be $\mathcal{G}^S$ and $H$ be planar.

Suppose we want to find a copy of $H$ in $R_n$ or verify there is no such subgraph. How quickly can we do so?

We see that we can do so in constant expected time; and similarly if we can seek an induced copy of $H$ or a minor $H$. 
Pendant copies theorem

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Smoothness

Let $\mathcal{A}$ be any small class of graphs.

Call $\mathcal{A}$ smooth if \[ \frac{|\mathcal{A}_n|}{n|\mathcal{A}_{n-1}|} \rightarrow \text{a limit as } n \rightarrow \infty. \]

In this case the limit must be the growth constant $\gamma$.

All the classes for which we know an asymptotic counting formula are smooth, for example series-parallel graphs, $\mathcal{P}$, $\mathcal{G}^S$.

Showing smoothness is an important step in proving results about $R_n \in_u \mathcal{A}$. 
When is $\mathcal{A}$ smooth?

Bender, Canfield and Richmond (2008):

**Theorem**

$G^S$ is smooth for any surface $S$.

The proof did not involve an asymptotic counting formula (and indeed none was then known). The method can be adapted to show more.

The **core** of $G$, core$(G)$, is the unique maximal subgraph such that the minimum degree $\delta(G) \geq 2$.

The idea is that, if the core grows reasonably smoothly then rooting trees in it yields a smooth class.

The proof method can be adapted to show that any addable minor-closed class is smooth, and indeed more generally.
Well-behaved graph classes

Our results will involve a well-behaved class of graphs $\mathcal{A}$. We require that $\mathcal{A}$ is proper, minor-closed and bridge-addable, and satisfies certain further conditions. The following classes of graphs are all well-behaved:

- any proper, minor-closed, addable class (for example the class of forests, or series-parallel graphs or planar graphs);
- the class $\mathcal{G}^S$ of graphs embeddable on any given surface $S$;
- the class of all graphs which contain at most $k$ vertex-disjoint cycles.

The definition of well-behaved requires $\mathcal{A}$ also to be ‘freely-addable-or-limited’.

[It suffices also for $\mathcal{A}$ to be closed under subdivision of edges, if there is a growth constant.]
Well-behaved graph classes
Freely-addable-or-limited classes of graphs

\( H \in \mathcal{A} \) is **freely addable** to \( \mathcal{A} \) if the disjoint union \( G \cup H \in \mathcal{A} \) whenever \( G \in \mathcal{A} \). (If \( \mathcal{A} \) is decomposable then each graph in \( \mathcal{A} \) is freely addable.)

\( H \in \mathcal{A} \) is **limited** in \( \mathcal{A} \) if \( kH \) is not in \( \mathcal{A} \) for some positive integer \( k \).

If \( \mathcal{A} \) is \( \mathcal{G}^S \) then the freely addable graphs are the planar graphs, and the limited graphs are the non-planar graphs in \( \mathcal{G}^S \).

\( \mathcal{A} \) is **freely-addable-or-limited** if each graph in \( \mathcal{A} \) is either freely addable or limited (it cannot be both).

Decomposable classes, \( \mathcal{G}^S \) and \( \text{Ex}(kC_3) \) are all freely-addable-or-limited.

\( \text{Ex}(C_3 \cup C_4) \) is **not** freely-addable-or-limited.
Smoothness and \( \text{core}(R_n) \) theorem

**Theorem**

Let \( \mathcal{A} \) be well-behaved, with growth constant \( \gamma > e \); let \( \mathcal{C} \) denote the class of connected graphs in \( \mathcal{A} \); and let \( R_n \in_u \mathcal{A} \). Then both \( \mathcal{A} \) and \( \mathcal{C} \) are smooth.

(a) \( \mathcal{C}^{\delta \geq 2} \) has growth constant \( \beta \) where \( \beta \) is the unique root \( > 1 \) to \( \beta e^{1/\beta} = \gamma \);

(b) Let \( \alpha = 1 - x \) where \( x \) is the unique root \( < 1 \) to \( xe^{-x} = 1/\beta \). Then

\[
\mathbb{P}( |v(\text{core}(R_n)) - \alpha n| > \epsilon n ) = e^{-\Omega(n)}.
\]

(b) Let \( \mathcal{D} \) denote the class of connected graphs freely addable to \( \mathcal{A} \). Let \( \rho = 1/\gamma \). Then \( T(\rho) < D(\rho) < \infty \), and the probability that \( \text{core}(R_n) \) is connected tends to \( e^{T(\rho) - D(\rho)} \).

Conjecture: every minor closed class is smooth?
Let us illustrate the last theorem for $G^S$. We have known since 2008 that $G^S$ has growth constant $\gamma$, where $\gamma$ is the \textit{planar graph growth constant}; and from Giménez and Noy (2009) we have $\gamma \approx 27.226878$.

[Counting $G^S$ was vastly improved in 2011 by Chapuy, Fusy, Giménez, Mohar and Noy, and by Bender and Gao, to give an asymptotic formula for $|G^S_n|$.

$G^S$ is well-behaved, and so $G^S$ is smooth, as we saw earlier – here we learn something new about the core.
Example: graphs on surfaces

\text{core}(R_n) \text{ for } R_n \in_u \mathcal{G}^S

Solving $\beta e^{1/\beta} = \gamma$ gives $\beta \approx 26.207554$. This is the growth constant of the class of (connected) graphs in $\mathcal{G}^S$ with minimum degree at least 2. The growth constant $\beta$ is only slightly larger than the growth constant $\approx 26.18412$ for 2-connected graphs in $\mathcal{G}^S$, from Bender, Gao and Wormald (2002).

Solving $\alpha = 1 - 1/\beta$ gives $\alpha \approx 0.961843$; and for $R_n \in_u \mathcal{G}^S$

$$v(\text{core}(R_n)) \sim \alpha n \text{ whp}$$

Also, the asymptotic number $\alpha n$ of vertices in the core of $R_n$ is only slightly larger than the number of vertices in the largest block of $R_n$, which is about $0.95982n$, from Giménez, Noy and Rué (2007). See also recent work of Noy and Ramos (2014).
Example: graphs on surfaces

$R_n \in \mathcal{U} \mathcal{G}^S$, connectivity of $\text{core}(R_n)$

Finally, consider the connectivity of the 2-core. The class $\mathcal{D}$ of connected freely addable graphs is the class of connected planar graphs. From Giménez and Noy (2009), $e^{-\mathcal{D}(\rho)} \approx 0.963253$ where $\rho = 1/\gamma$.

Further $e^{\mathcal{T}(\rho)} \approx 1.038138$, so by part (d) of the last Theorem, the probability that $\text{core}(R_n)$ is connected $\approx 0.999990$.

Thus

$$\mathbb{P}(\text{core}(R_n) \text{ not connected}) \approx 10^{-5}.$$ 

For comparison

$$\mathbb{P}(\text{Frag}(R_n) = C_3) \sim e^{-\mathcal{D}(\rho)} \rho^3/6 \approx 8 \cdot 10^{-6}.$$
Boltzmann Poisson random graph

Let $\mathcal{A}$ be decomposable.

Fix $\rho > 0$ such that $A(\rho)$ is finite; and let

$$\mu(H) = \frac{\rho^{v(H)}}{\text{aut}(H)} \quad \text{for each } H \in \mathcal{UA}.$$

Here $\mathcal{UA}$ denotes the unlabelled graphs in $\mathcal{A}$. Easy to see:

$$A(\rho) = \sum_{H \in \mathcal{UA}} \mu(H).$$
Boltzmann Poisson random graph

The Boltzmann Poisson random graph $R = R(\mathcal{A}, \rho)$ takes values in $\mathcal{UA}$, with

$$\mathbb{P}[R = H] = \frac{\mu(H)}{A(\rho)} \quad \text{for each } H \in \mathcal{UA}. $$

Let $\mathcal{C}$ denote the class of connected graphs in $\mathcal{A}$. For each $H \in \mathcal{UC}$ let $\kappa(G, H)$ be the number of components of $G$ isomorphic to $H$.

**Proposition**

The random variables $\kappa(R, H)$ for $H \in \mathcal{UC}$ are independent, with $\kappa(R, H) \sim \text{Po}(\mu(H))$. 
Fragments theorem

Recall that $H \in \mathcal{A}$ is freely addable to $\mathcal{A}$ if the disjoint union $G \cup H \in \mathcal{A}$ whenever $G \in \mathcal{A}$.

Let $\mathcal{F}_\mathcal{A}$ be the class of graphs freely addable to $\mathcal{A}$. Observe that $\mathcal{F}_\mathcal{A}$ is decomposable. Also, if $\mathcal{A}$ is bridge-addable then so is $\mathcal{F}_\mathcal{A}$, and then $\mathcal{F}_\mathcal{A}$ is addable.

**Theorem**

Let $\mathcal{A}$ be well-behaved, and let $\rho = \rho(\mathcal{A})$. Let $\mathcal{F}_\mathcal{A}$ be the class of graphs freely addable to $\mathcal{A}$, with egf $F_{\mathcal{A}}$.

Then $0 < \rho < \infty$ and $F_{\mathcal{A}}(\rho)$ is finite;

and for $R_n \in \mathcal{A}$, $F_n = \mathcal{U}\text{Frag}(R_n)$ satisfies $F_n \to_d R$ where $R$ is the Boltzmann Poisson random graph $R(\mathcal{F}_\mathcal{A}, \rho)$. 
Corollaries on Fragments and connectivity

**Corollary**

Let $\mathcal{D}$ be the class of connected graphs in $\mathcal{F}_A$. Given distinct graphs $H_1, \ldots, H_k$ in $\mathcal{UD}$ the $k$ random variables $\kappa(F_n, H_i)$ are asymptotically independent with distribution $\text{Po}(\mu(H_i))$.

This gives for example, for $R_n \in u A$

$$\mathbb{P}(R_n \text{ is connected }) \to e^{-D(\rho)}.$$  

Consider trees $\mathcal{T}$ and forests $\mathcal{F}$, where $\rho = 1/e$. For $R_n \in u \mathcal{F}$, since $T(\rho) = \frac{1}{2}$,

$$\mathbb{P}(R_n \text{ is connected }) = \frac{|T_n|}{|F_n|} \to e^{-\frac{1}{2}}.$$
Other minor-closed classes
Connected excluded minors

What behaviour can we see with other minor-closed classes, not well-behaved?

Example
Path forests, ie $\text{Ex}(C_3, K_{1,3})$.
Decomposable but not bridge-addable.
Smooth with growth constant 1.
$\kappa(R_n)$ asymptotically normal, mean $\sim \sqrt{n}$.
Largest component size $\sim \sqrt{n \log n}$.

More examples in:
M. Bousquet-Mélou and K. Weller (2014) Asymptotic properties of some minor-closed classes of graphs
Other minor-closed classes
Disconnected excluded minors

**Example**
At most $k$ disjoint cycles, ie $Ex((k+1)C_3)$.
Not decomposable.
Looks like a forest with $k$ additional ‘free’ vertices.
Smooth with growth constant $\gamma_k = 2^k e$.

$P(R_n)$ is connected $\rightarrow p_k := e^{-T(1/\gamma_k)}$ ($p_0 = e^{-\frac{1}{2}}$).

Similar behaviour for $Ex((k+1)C_t)$, $Ex((k+1)D)$, $Ex((k+1)K_{1,t})$,... (and for unlabelled graphs with few disjoint cycles) but not for $Ex(2K_4)$.

Disjoint excluded minors

disjoint minors from an addable class

Let \( \text{apex}^k \mathcal{A} \) denote the set of \( G \) such that there is a set \( X \) of at most \( k \) vertices with \( G - X \in \mathcal{A} \).

A fan is a path \( P \) together with a vertex adjacent to each vertex on \( P \).

Let \( \mathcal{A} \) be addable, with set \( \mathcal{B} \) of (2-connected) excluded minors (so \( \mathcal{A} = \text{Ex}(\mathcal{B}) \)).

If \( \mathcal{A} \) does not contain all fans, then \( \text{Ex}(k + 1)\mathcal{B} \) is the union of \( \text{apex}^k \mathcal{A} \) and an exponentially smaller class.

If \( \mathcal{A} \) contains all fans (eg \( \mathcal{A} = \text{Ex}(K_4) \)) then this is false.

[\( \text{Ex}(K_{1,t}) \) is not addable. Disjoint \( K_{1,t} \) minors behave a little differently for \( t \geq 4 \): the difference class is smaller by a factor \( 2^{\Theta(\frac{2t-5}{2t-4})} \).]
Disjoint excluded minors

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[$\text{Ex}(K_1,t)$ is not addable. Disjoint $K_1,t$ minors behave a little differently for $t \geq 4$: the difference class is smaller by a factor $2^{-\Theta(n^{\frac{2t-5}{2t-4}})}$. ]
Disjoint $K_4$ minors 1

What about $\text{Ex}(2K_4)$?

$\text{Ex}(K_4)$ has growth constant $\gamma \approx 9.07$ (Bernardi, Noy and Welsh 2010). So $\text{apex}(\text{Ex}(K_4))$ has growth constant $2\gamma$.

$\text{Ex}(2K_4) \supseteq \text{apex}^3 \mathcal{F}$.

$\text{apex}^3 \mathcal{F}$ has growth constant $2^3 e > 2\gamma$.

Thus $\text{apex}(\text{Ex}(K_4))$ is exponentially smaller than $\text{Ex}(2K_4)$. 
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$\text{Ex}(2K_4) \supseteq \text{apex}^3 \mathcal{F}$.

$\text{apex}^3 \mathcal{F}$ has growth constant $2^3 \epsilon > 2\gamma$.

Thus $\text{apex}((\text{Ex}(K_4))$ is exponentially smaller than $\text{Ex}(2K_4)$.
Disjoint $K_4$ minors 2

A model of $H$ in $G$ is a subgraph of $G$ contractible to $H$.

Observe: $\text{apex}^kA$ is the set of $G$ such that there is a set $X$ of at most $k$ vertices satisfying $|X \cap V(B)| \geq 1$ for each model $B$ of a graph in $\mathcal{B}$. Name it again as $1BL^k(B)$.

Let $jBL^k(B)$ denote the set of $G$ such that there is a set $X$ of at most $k$ vertices satisfying $|X \cap V(B)| \geq j$ for each model $B$ of a graph in $\mathcal{B}$.

\[
\text{apex}^3 \mathcal{F} \subseteq 2BL^3K_4 \subseteq \text{Ex}(2K_4).
\]
Disjoint $K_4$ minors 3

Kurauskas and McD (2012) conjectured that $\text{Ex}(2K_4)$ is the union of $2BL^3(K_4)$ and an exponentially smaller class; and more generally $\text{Ex}((k + 1)K_4)$ is the union of $2BL^{2k+1}(K_4)$ and an exponentially smaller class.

This has recently been proved by Valentas Kurauskas (2013).

For each $j$ there are graphs $H$ such that $\text{Ex}(2H)$ is the union of $jBL^{2j-1}(H)$ and an exponentially smaller class, and no smaller $j'$ works....
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General model

Binomial random graph $G_{n,p}$

In the classical binomial random graph $G_{n,p}$ on the vertex set $[n]$, the $\binom{n}{2}$ possible edges appear independently with probability $p$, $0 < p < 1$.

For each $H \in \mathcal{A}_n$

$$\mathbb{P}(G_{n,p} = H \mid G_{n,p} \in \mathcal{A}) = \frac{p^{e(H)}(1 - p)^{\binom{n}{2} - e(H)}}{\sum_{G \in \mathcal{A}_n} p^{e(G)}(1 - p)^{\binom{n}{2} - e(G)}} = \frac{\lambda^{e(H)}}{\sum_{G \in \mathcal{A}_n} \lambda^{e(G)}}$$

where $\lambda = p/(1 - p)$.

Here we assume that $\mathcal{A}_n \neq \emptyset$, and $e(G)$ denotes the number of edges in $G$. 
General model
Random cluster model

In the more general random-cluster model, we are also given $\nu > 0$; and the random graph $R_n$ ranges over the graphs $H$ on $[n]$, with

$$\mathbb{P}(R_n = H) \propto p^{e(H)}(1 - p)^{\binom{n}{2} - e(H)} \cdot \nu^{\kappa(H)}.$$ 

Here $\kappa(H)$ denotes the number of components of $H$. For each $H \in \mathcal{A}_n$ we have

$$\mathbb{P}(R_n = H \mid R_n \in \mathcal{A}) = \frac{\lambda^{e(H)} \nu^{\kappa(H)}}{\sum_{G \in \mathcal{A}_n} \lambda^{e(G)} \nu^{\kappa(G)}}.$$
General model

New model

The distribution of our random graphs in $\mathcal{A}$ is as follows.

Given edge-parameter $\lambda > 0$ and component-parameter $\nu > 0$, we let the weighting $\tau$ be the pair $(\lambda, \nu)$. For each graph $G$ we let

$$\tau(G) = \lambda^{e(G)} \nu^{\kappa(G)};$$

and we denote $\sum_{G \in \mathcal{A}_n} \tau(G)$ by $\tau(\mathcal{A}_n)$.

$R_n \in_\tau \mathcal{A}$ means that $R_n$ is a random graph which takes values in $\mathcal{A}_n$ with

$$\mathbb{P}(R_n = H) = \frac{\tau(H)}{\tau(\mathcal{A}_n)}.$$

We call $R_n$ a $\tau$-weighted random graph from $\mathcal{A}$.

When $\lambda = \nu = 1$ we are back to random graphs sampled uniformly. Think of this case!